

Radiation of ELF/VLF waves by harmonically varying currents into a stratified ionosphere, with application to radiation by a modulated electrojet

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We use a full-wave finite element approach inspired by the mode calculation method of *Wait* [1970] to calculate electromagnetic fields in a horizontally-stratified ionosphere treated as a magnetized plasma. The source currents can have arbitrary vertical and horizontal distributions. The electromagnetic field is calculated both in the Earth-ionosphere waveguide (at arbitrary horizontal distance and direction) and in the ionosphere (as an upward propagating whistler mode wave). This method requires less computational resources than traditional FDFD and FDTD methods, while providing results with similar accuracy. The model is applied to the radiation of ELF/VLF waves from modulated HF-heated electrojet currents, and the calculated values are compared to ground and satellite observations.

1. Introduction

There is a plethora of methods for finding fields in stratified media [*Wait*, 1970; *Bossy*, 1979; *Yagitani et al.*, 1994; *Nygrén*, 1981; *Nagano et al.*, 1994], which are summarized, e.g., by *Budden* [1985, ch. 18]. However, many of them have difficulty with numerical stability when the evanescent wave solutions (with a large imaginary component of the vertical wave number) “swamp” the waves of interest, which requires special tricks to contain the stability [*Pitteway*, 1965; *Nygrén*, 1982]. A detailed discussion of this effect can be found in *Budden* [1985, p. 574–576]. Here we describe a method which is inherently stable against such “swamping”, which is based on the idea of recursive calculation of reflection coefficients and mode amplitudes by *Nygrén* [1982] and which also allows the use of an arbitrary configuration of the radiating sources.

2. Description of the method

2.1. Waves in stratified medium

Consider electromagnetic waves with time dependence $\sim e^{-i\omega t}$ in a horizontally stratified ionosphere. Let us choose a coordinate system with a vertical upward z -axis (perpendicular to the layers) and x , y axes in the horizontal plane. We divide the medium into layers with boundaries z_k , $k=1 \dots M$ [*Wait*, 1970], with $z_1=0$ corresponding to the ground. In each layer $\Omega_k = (z_k, z_{k+1})$ (where $k=1 \dots M-1$) and $\Omega_M = (z_M, \infty)$, the medium is assumed to be uniform, with permeability $\mu=1$ and permittivity (at given frequency ω) described by the tensor $\hat{\epsilon}_k$. The thickness of layer Ω_k is $h_k = z_{k+1} - z_k$, $k=1 \dots M-1$.

Maxwell’s equations for a source-free medium are

$$\nabla \times \mathbf{H} = -ik_0 \hat{\epsilon} \mathbf{E} \quad (1)$$

$$\nabla \times \mathbf{E} = ik_0 \mathbf{H} \quad (2)$$

where $k_0 = \omega/c$ and for convenience we use SI units for \mathbf{E} and modified units for \mathbf{H} , so that $\mathbf{H} = Z_0 \mathbf{H}_{\text{SI}}$ [*Budden*, 1985, p. 31], where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space.

We seek a solution in a form of a linear combination of plane waves $\sim e^{ik_0(\mathbf{n} \cdot \mathbf{r})}$ in each layer, where $k_0 = \omega/c$, and \mathbf{n} is the refractive index vector, whose horizontal component \mathbf{n}_\perp is conserved by Snell’s law, so that $n_x = \text{const}$, $n_y = \text{const}$. Inside each layer, we have

$$\mathbf{n} \times \mathbf{H} = -\hat{\epsilon} \mathbf{E} \quad (3)$$

$$\mathbf{n} \times \mathbf{E} = \mathbf{H} \quad (4)$$

Substituting the second equation into the first, we find an equation for \mathbf{E}

$$(n^2 \mathbf{I} - \mathbf{nn}^T - \hat{\epsilon}) \mathbf{E} = 0 \quad (5)$$

and the dispersion equation

$$\det(n^2 \mathbf{I} - \mathbf{nn}^T - \hat{\epsilon}) = 0 \quad (6)$$

where \mathbf{I} is a unit tensor and \mathbf{nn}^T is a diadic product.

The dispersion equation (6), when written as an equation for n_z , is a quartic equation which is usually called the Booker quartic [*Booker*, 1938]. We use a standard method to solve it [e.g., *Abramowitz and Stegun*, 1970, p. 17]. In general, this equation has 4 roots $n_z^{(m)}$, $m=1 \dots 4$, with a possibility of multiple roots. We sort the 4 solutions obtained for n_z according to their imaginary part. A positive imaginary part means that the wave is evanescent or absorbed in the upward direction, and negative means that the wave is evanescent or absorbed in the downward direction. Based on this reasoning, we label the 2 modes corresponding to the higher imaginary parts of n_z as upward-propagating, and the other 2 modes as downward-propagating, and denote $n_z^{u1} \equiv n_z^{(1)}$, $n_z^{u2} \equiv n_z^{(2)}$, $n_z^{d1} \equiv -n_z^{(3)}$, $n_z^{d2} \equiv -n_z^{(4)}$, so that all n_z^{ui} , n_z^{di} , $i=1,2$, have non-negative imaginary parts.

The field structure of each mode (components of $\mathbf{E}^{(m)}$, $\mathbf{H}^{(m)}$) is found from (5) and (4), with arbitrary normalization. We can choose, e.g., $|\mathbf{E}^{(m)}|^2 = 1$. These components make up a 6×4 matrix, which is constant in each layer Ω_k . We denote this matrix \mathcal{F}_k , with components $(\mathcal{F}_k)_{1,m} = E_x^{(m)}$, $(\mathcal{F}_k)_{2,m} = E_y^{(m)}$ etc. The fields are obtained using these matrices from the upward- and downward-propagating mode amplitudes $\mathbf{u}(z)$ and $\mathbf{d}(z)$, respectively:

$$\begin{pmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{pmatrix} = \mathcal{F}_k \begin{pmatrix} \mathbf{u}(z) \\ \mathbf{d}(z) \end{pmatrix} e^{ik_0(\mathbf{n}_\perp \cdot \mathbf{r}_\perp)} \quad \text{in layer } \Omega_k \quad (7)$$

where \mathbf{u} and \mathbf{d} are arrays of length 2:

$$\mathbf{u} = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} d^1 \\ d^2 \end{pmatrix} \quad (8)$$

For example, in an isotropic medium ($\hat{\varepsilon} \equiv \varepsilon$, $n_x^2 + n_z^2 = n^2 = \varepsilon$) we can choose the TE mode for u^1 and d^1 and the TM mode for u^2 and d^2 :

$$\mathcal{F}_{\text{isotropic}} = \begin{pmatrix} -\sin \phi & (n_z/n) \cos \phi & -\sin \phi & (n_z/n) \cos \phi \\ \cos \phi & (n_z/n) \sin \phi & \cos \phi & (n_z/n) \sin \phi \\ 0 & -n_{\perp}/n & 0 & n_{\perp}/n \\ -n_z \cos \phi & -n \sin \phi & n_z \cos \phi & n \sin \phi \\ -n_z \sin \phi & n \cos \phi & n_z \sin \phi & -n \cos \phi \\ n_{\perp} & 0 & n_{\perp} & 0 \end{pmatrix} \quad (9)$$

where $\cos \phi = n_x/n_{\perp}$, $\sin \phi = n_y/n_{\perp}$.

We are interested mostly in the horizontal components of \mathbf{E} , \mathbf{H} , since they have to be continuous at the boundaries between layers in the absence of sources. These components are given by a 4×4 matrix \mathbf{F}_k which is a submatrix of \mathcal{F}_k :

$$\begin{pmatrix} \mathbf{E}_{\perp}(z) \\ \mathbf{H}_{\perp}(z) \end{pmatrix} = \mathbf{F}_k \begin{pmatrix} \mathbf{u}(z) \\ \mathbf{d}(z) \end{pmatrix} \quad \text{in layer } \Omega_k \quad (10)$$

(we do not include the factor $e^{ik_0(\mathbf{n}_{\perp} \cdot \mathbf{r}_{\perp})}$ in \mathbf{E}_{\perp} , \mathbf{H}_{\perp}). In each layer Ω_k , $\mathbf{u}(z)$ and $\mathbf{d}(z)$ are given in terms of their values at $z_k + 0$ (i.e., just above the boundary z_k), which we denote as \mathbf{u}_k and \mathbf{d}_k :

$$u^i(z) = u_k^i e^{ik_0 n_z^{ui}(z-z_k)} \quad (11)$$

$$d^i(z) = d_k^i e^{-ik_0 n_z^{di}(z-z_k)}, \quad i = 1, 2 \quad (12)$$

At $z = z_{k+1} - 0$ (still in the layer Ω_k , just below the boundary z_{k+1}), we have

$$\mathbf{u}'_{k+1} \equiv \mathbf{u}(z_{k+1}) = \mathbf{P}_k^u \mathbf{u}_k \quad (13)$$

$$\mathbf{d}'_{k+1} \equiv \mathbf{d}(z_{k+1}) = (\mathbf{P}_k^d)^{-1} \mathbf{d}_k \quad (14)$$

where \mathbf{P}^u , \mathbf{P}^d describe propagation in a uniform medium over layer Ω_k :

$$\mathbf{P}^u = \begin{pmatrix} e^{ik_0 n_z^{u1} h_k} & 0 \\ 0 & e^{ik_0 n_z^{u2} h_k} \end{pmatrix} \quad (15)$$

$$\mathbf{P}^d = \begin{pmatrix} e^{ik_0 n_z^{d1} h_k} & 0 \\ 0 & e^{ik_0 n_z^{d2} h_k} \end{pmatrix} \quad (16)$$

Due to the way we chose n_z^{ui} , n_z^{di} to have non-negative imaginary parts, the matrices \mathbf{P}^u , \mathbf{P}^d have bounded norms (the amplitude of the wave is non-increasing as it propagates), a fact which allows us to devise a stable method of finding the fields.

The fields \mathbf{E}_{\perp} , \mathbf{H}_{\perp} are continuous at each boundary z_k , $k = 2 \dots M$ (in the absence of sources). Therefore, we have a boundary condition

$$\mathbf{F}_k \begin{pmatrix} \mathbf{u}'_{k+1} \\ \mathbf{d}'_{k+1} \end{pmatrix} = \mathbf{F}_{k+1} \begin{pmatrix} \mathbf{u}_{k+1} \\ \mathbf{d}_{k+1} \end{pmatrix} \quad (17)$$

From these equations, we can find matrices for transport through boundary z_{k+1} :

$$\begin{pmatrix} \mathbf{u}'_{k+1} \\ \mathbf{d}'_{k+1} \end{pmatrix} = \mathbf{T}_k^d \begin{pmatrix} \mathbf{u}_{k+1} \\ \mathbf{d}_{k+1} \end{pmatrix} \quad (18)$$

$$\begin{pmatrix} \mathbf{u}_{k+1} \\ \mathbf{d}_{k+1} \end{pmatrix} = \mathbf{T}_k^u \begin{pmatrix} \mathbf{u}'_{k+1} \\ \mathbf{d}'_{k+1} \end{pmatrix} \quad (19)$$

$$(20)$$

where \mathbf{T}_k^u and \mathbf{T}_k^d are inverses of each other and are given by

$$\mathbf{T}_k^d = \mathbf{F}_k^{-1} \mathbf{F}_{k+1} \quad (21)$$

$$\mathbf{T}_k^u = \mathbf{F}_{k+1}^{-1} \mathbf{F}_k \quad (22)$$

If the medium does not change at the boundary z_{k+1} , then \mathbf{T}_k^u and \mathbf{T}_k^d should be unit matrices.

2.2. Reflection coefficients

Introduce \mathbf{R}_k^u and \mathbf{R}_k^d as the reflection coefficients for waves travelling up and down, respectively, so that

$$\mathbf{d}_k = \mathbf{R}_k^u \mathbf{u}_k \quad (23)$$

if all sources are located below z_k , and

$$\mathbf{u}_k = \mathbf{R}_k^d \mathbf{d}_k \quad (24)$$

if all sources are located above z_k . We now demonstrate that there is no ‘‘swamping’’ instability if \mathbf{R}_k^u is calculated entirely in terms of \mathbf{R}_{k+1}^u , and \mathbf{R}_{k+1}^d is calculated entirely in terms of \mathbf{R}_k^d .

Let us consider a wave propagating upward which reflects at boundary z_{k+1} with a known reflection coefficient \mathbf{R}_{k+1}^u , so that $\mathbf{d}_{k+1} = \mathbf{R}_{k+1}^u \mathbf{u}_{k+1}$. Then, using also (14), we find that the downward mode amplitude at boundary z_k is

$$\begin{aligned} \mathbf{d}_k &= \mathbf{P}_k^d \mathbf{d}'_{k+1} = \mathbf{P}_k^d (\mathbf{T}_{k,du}^d \mathbf{u}_{k+1} + \mathbf{T}_{k,dd}^d \mathbf{d}_{k+1}) \\ &= \mathbf{P}_k^d (\mathbf{T}_{k,du}^d + \mathbf{T}_{k,dd}^d \mathbf{R}_{k+1}^u) \mathbf{u}_{k+1} \end{aligned} \quad (25)$$

where we used equation (18) written in a block-matrix form, so that $\mathbf{T}_{k,du}^d$ and $\mathbf{T}_{k,dd}^d$ are 2×2 submatrices of \mathbf{T}_k^d . On the other hand, $\mathbf{d}_k = \mathbf{R}_k^u \mathbf{u}_k$ and

$$\begin{aligned} \mathbf{u}_k &= (\mathbf{P}_k^u)^{-1} \mathbf{u}'_{k+1} = (\mathbf{P}_k^u)^{-1} (\mathbf{T}_{k,uu}^d \mathbf{u}_{k+1} + \mathbf{T}_{k,ud}^d \mathbf{d}_{k+1}) \\ &= (\mathbf{P}_k^u)^{-1} (\mathbf{T}_{k,uu}^d + \mathbf{T}_{k,ud}^d \mathbf{R}_{k+1}^u) \mathbf{u}_{k+1} \end{aligned} \quad (26)$$

From these two results, since they should hold for arbitrary \mathbf{u}_{k+1} , we obtain the recursive relation for \mathbf{R}_k^u , the reflection coefficient from above:

$$\mathbf{R}_k^u = \mathbf{P}_k^d (\mathbf{T}_{k,du}^d + \mathbf{T}_{k,dd}^d \mathbf{R}_{k+1}^u) (\mathbf{T}_{k,uu}^d + \mathbf{T}_{k,ud}^d \mathbf{R}_{k+1}^u)^{-1} \mathbf{P}_k^u \quad (27)$$

Analogously, we obtain a recursive relation for the reflection coefficient from below:

$$\mathbf{R}_{k+1}^d = (\mathbf{T}_{k,uu}^u \mathbf{P}_k^u \mathbf{R}_k^d \mathbf{P}_k^d + \mathbf{T}_{k,ud}^u) (\mathbf{T}_{k,du}^u \mathbf{P}_k^u \mathbf{R}_k^d \mathbf{P}_k^d + \mathbf{T}_{k,dd}^u)^{-1} \quad (28)$$

Both of these equations are stable when n_z have large imaginary parts because the matrices \mathbf{P}^u , \mathbf{P}^d are bounded. The recursive formulas (27) and (28) must use initial values for \mathbf{R}_M^u and \mathbf{R}_1^d . We assume that the upper boundary is free, so we have $\mathbf{R}_M^u = 0$. For a conducting Earth, we have $\mathbf{E}_{\perp} = 0$ at $z = z_1$. Substituting $\mathbf{E}_{\perp} = 0$ and $\mathbf{u}_1 = \mathbf{R}_1^d \mathbf{d}_1$ into (10), we find that $\mathbf{R}_1^d = -(\mathbf{F}_{1,Eu})^{-1} \mathbf{F}_{1,Ed}$, where, e.g., $\mathbf{F}_{k,Eu}$ is a 2×2 submatrix of \mathbf{F}_k with rows corresponding to \mathbf{E}_{\perp} and columns corresponding to \mathbf{u} . In an isotropic medium, using (9) we obtain $\mathbf{R}_1^d = -1$.

2.3. Calculation of mode amplitudes

Now, let us derive recursive relations for \mathbf{u} above the sources and \mathbf{d} below the sources (the mode amplitudes for the other direction are obtained using the reflection coefficients, see Equations (23–24)). Let us express \mathbf{u}_{k+1} in terms of \mathbf{u}_k , using (19) and (13–14):

$$\begin{aligned}\mathbf{u}_{k+1} &= \mathbf{T}_{k,uu}^u \mathbf{u}'_{k+1} + \mathbf{T}_{k,ud}^u \mathbf{d}'_{k+1} \\ &= \mathbf{T}_{k,uu}^u \mathbf{P}_k^u \mathbf{u}_k + \mathbf{T}_{k,ud}^u (\mathbf{P}_k^d)^{-1} \mathbf{d}_k \\ &= (\mathbf{T}_{k,uu}^u \mathbf{P}_k^u + \mathbf{T}_{k,ud}^u (\mathbf{P}_k^d)^{-1} \mathbf{R}_k^u) \mathbf{u}_k\end{aligned}\quad (29)$$

Thus, we have the recursive relation sought

$$\mathbf{u}_{k+1} = \mathbf{U}_k \mathbf{u}_k \quad (30)$$

where

$$\mathbf{U}_k = \mathbf{T}_{k,uu}^u \mathbf{P}_k^u + \mathbf{T}_{k,ud}^u (\mathbf{P}_k^d)^{-1} \mathbf{R}_k^u \quad (31)$$

There is no potential instability in (31) from $(\mathbf{P}_k^d)^{-1}$ because it cancels with \mathbf{P}_k^d in the expression for the reflection coefficient (27). Analogously, we find that below the sources

$$\mathbf{d}_k = \mathbf{D}_k \mathbf{d}_{k+1} \quad (32)$$

where

$$\mathbf{D}_k = \mathbf{P}_k^d (\mathbf{T}_{k,du}^d \mathbf{R}_{k+1}^d + \mathbf{T}_{k,dd}^d) \quad (33)$$

which is also stable.

2.4. Dealing with the sources

Assume that currents have harmonic dependence on time and perpendicular coordinates, i.e., $\sim e^{ik_0(\mathbf{n}_\perp \cdot \mathbf{r}_\perp) - i\omega t}$ and flow only in the plane $z = z_s + 0$, just above the boundary between layers Ω_{s-1} and Ω_s , so that they are still inside Ω_s :

$$\mathbf{J}(\mathbf{r}, t) = (\mathbf{J}_\perp + \hat{z} J_z) \delta(z - z_s) e^{ik_0(\mathbf{n}_\perp \cdot \mathbf{r}_\perp) - i\omega t} \quad (34)$$

Our task is to find the wave amplitudes $\mathbf{u}(z)$ and $\mathbf{d}(z)$ inside layer Ω_s just above (\mathbf{u}_s^+ , \mathbf{d}_s^+) and just below (\mathbf{u}_s^- , \mathbf{d}_s^-) the source plane.

The fields satisfy boundary conditions

$$\Delta \mathbf{E}_\perp \equiv \mathbf{E}_s^+ - \mathbf{E}_s^- = Z_0 J_z \mathbf{n}_\perp \quad (35)$$

$$\Delta \mathbf{H}_\perp \equiv \mathbf{H}_s^+ - \mathbf{H}_s^- = (Z_0 \mathbf{J}_\perp \times \hat{z}) \quad (36)$$

The first condition is derived in Appendix A, while the second condition is derived by integration of one of the Maxwell's equations across the boundary. The change $\Delta \mathbf{u}$ and $\Delta \mathbf{d}$ at the plane of the source is obtained using (10):

$$\begin{pmatrix} \Delta \mathbf{u} \\ \Delta \mathbf{d} \end{pmatrix} = \mathbf{F}_s^{-1} \begin{pmatrix} \Delta \mathbf{E}_\perp \\ \Delta \mathbf{H}_\perp \end{pmatrix} \quad (37)$$

Since there are no sources other than at z_s , we can use reflection coefficients (23) and (24) to relate \mathbf{u}_s^\pm and \mathbf{d}_s^\pm :

$$\mathbf{d}_s^+ = \mathbf{R}_s^u \mathbf{u}_s^+ \quad (38)$$

$$\mathbf{u}_s^- = \mathbf{R}_s^d \mathbf{d}_s^- \quad (39)$$

Substituting

$$\mathbf{u}_s^- = \mathbf{u}_s^+ - \Delta \mathbf{u} \quad (40)$$

$$\mathbf{d}_s^- = \mathbf{d}_s^+ - \Delta \mathbf{d} \quad (41)$$

we find the solution

$$\mathbf{u}_s^+ = (1 - \mathbf{R}_s^d \mathbf{R}_s^u)^{-1} (\Delta \mathbf{u} - \mathbf{R}_s^d \Delta \mathbf{d}) \quad (42)$$

$$\mathbf{d}_s^+ = \mathbf{R}_s^u \mathbf{u}_s^+ \quad (43)$$

Now we can find all \mathbf{u}_k and \mathbf{d}_k at all altitudes, using (30) and (32), e.g., $\mathbf{u}_{s+1} = \mathbf{U}_s \mathbf{u}_s^+$, $\mathbf{d}_{s-1} = \mathbf{D}_{s-1} \mathbf{d}_s^-$ etc. Note that (42) does not give a valid answer if $\det(1 - \mathbf{R}_s^d \mathbf{R}_s^u) = 0$, which is the resonance condition. However, since there are

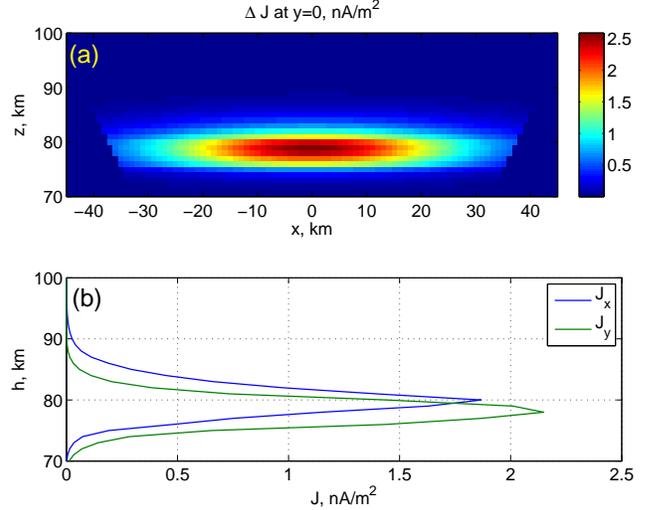


Figure 1. (a) Absolute value of the amplitude of modulated current density at $y = 0$. The vertical currents are negligible, the horizontal currents are due to Pedersen and Hall mechanisms. (b) The components of the current density at $x = 0$, $y = 0$. The electrojet ambient field is $E_0 = \hat{x}(-25 \text{ mV})$, so J_x corresponds to Pedersen and J_y to Hall current.

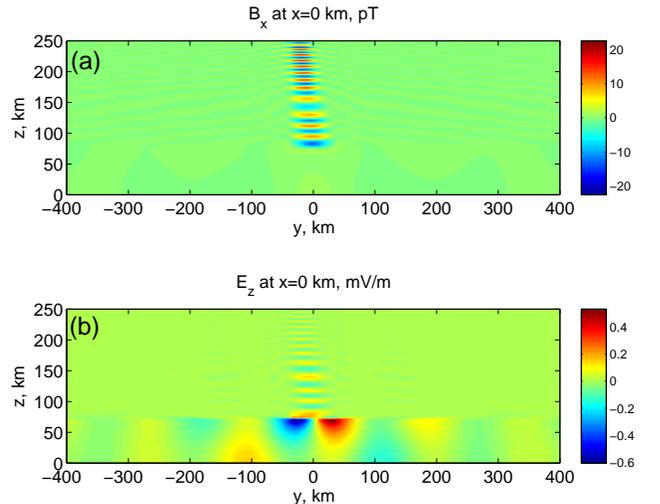


Figure 2. Fields in the $x = 0$ plane: (a) B_x ; (b) E_z . The upper panel illustrates the formation of the whistler “funnel”, and the lower panel the propagation in the Earth-ionosphere waveguide.

always some losses in the medium, the resonance condition cannot be satisfied for real \mathbf{n}_\perp .

If the sources occupy more than one layer, we can find the total field by summing fields created by each layer. For arbitrary horizontal profiles of the sources, we use Fourier decomposition of the sources, and then an inverse transform to calculate the fields. Using a discrete Fourier decomposition forces us to have periodic boundary conditions at the side boundaries. We should choose these boundaries far enough so that the waves are attenuated in the horizontal direction to the degree that the sources in the next horizontal “period” do not change the results in the computational domain. The method can be applied both in 2D and in 3D. In the latter case, the Fourier decomposition is over two horizontal components of \mathbf{n}_\perp .

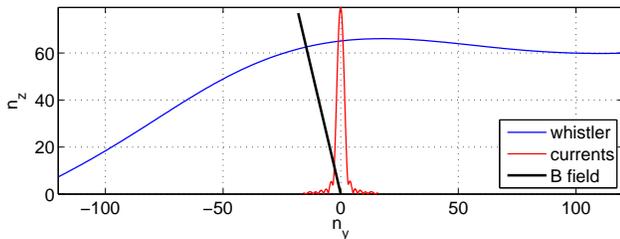


Figure 3. The refraction coefficient at the satellite altitude of $h = 700$ km, and the horizontal distribution of the Fourier components of the currents (arbitrary absolute scale). Both are taken at $n_x = 0$.

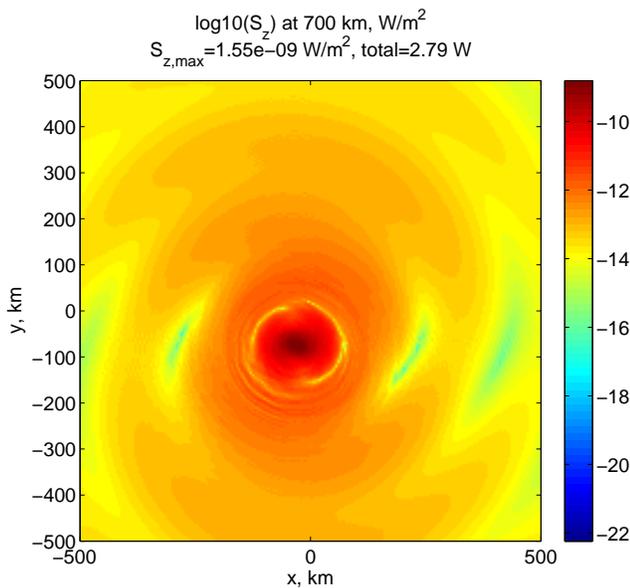


Figure 4. The calculated vertical Poynting vector at $h = 700$ km.

3. Application to the problem of a modulated electrojet in an HF-heated ionosphere

The magnetized plasma dielectric tensor for geomagnetic $\mathbf{B}_0 \parallel \hat{z}$ is [Stix, 1992]

$$\hat{\epsilon}_z = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \quad (44)$$

In a general case of an arbitrary direction of \mathbf{B}_0 , the dielectric tensor $\hat{\epsilon}$ is obtained by rotating $\hat{\epsilon}_z$ from an appropriately chosen coordinate system in which $z' \parallel \mathbf{B}_0$ to a system with vertical \hat{z} .

As an example, we calculate the fields created by electrojet currents modulated at frequency of 1875 Hz by the HAARP HF heating facility (as of April 2003). The change in the electrojet currents is calculated by Payne *et al.* [2007], for heating of the ionosphere by a sinusoidally modulated fully-focused HF beam at 3.2 MHz and effective radiated power (ERP) of ~ 24 MW. The x -axis is chosen in the eastward direction, and y -axis in the northward direction. The modulated currents are horizontal, and occupy a volume which has a shape of a horizontal pancake, with a slice shown in Figure 1a and the profile at the central vertical axis shown in Figure 1b. The altitude-dependent geomagnetic field is calculated using the IGRF model [http://modelweb.gsfc.nasa.gov/models/igrf.html] for the HAARP site, and has components $\mathbf{B}_0 \simeq \{6 \times 10^{-6}, 1.3 \times 10^{-5}, -5.3 \times 10^{-5}\}$ T at the altitude of the D -region currents. The electron density and collision frequency profiles are assumed to be the same as in [Payne *et al.*, 2007], with electron density profile extended to higher altitudes using the IRI model [http://modelweb.gsfc.nasa.gov/models/iri.html]. A vertical slice of two components of calculated fields is shown in Figure 2 and is discussed below in regard to satellite and ground-based observations.

3.1. Upward propagating fields and comparison with satellite observations

Due to the large horizontal size of the region occupied by the radiating currents $L \simeq 50$ km (Figure 1a), the horizontal components of the refractive index $n_\perp = k_\perp/k_0 \simeq (k_0 L)^{-1} \lesssim 1$ of radiated waves is not very large, especially compared to the refractive index of the whistler mode at the satellite altitudes (~ 700 km). This fact is illustrated in Figure 3. Due to this, the whistler-mode “beam” which is radiated upward does not have much lateral spread, and forms a narrow “funnel” (see Figures 2a and 4).

The vertical energy flux (Poynting vector) $S_z = (2Z_0)^{-1} \text{Re}[(\mathbf{E}_\perp)_M \times (\mathbf{H}_\perp)_M]_z$ calculated at $h = 700$ km is shown in Figure 4. The maximum fields are $E_{\max} \simeq$

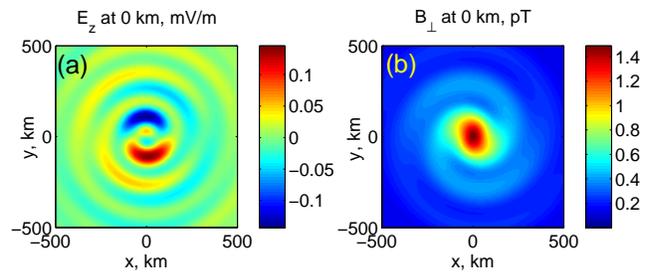


Figure 5. The calculated fields at $h = 0$ km: (a) E_z ; (b) B_\perp .

150 $\mu\text{V m}^{-1}$ and $B_{\text{max}} \simeq 30$ pT and occupy a spot of the size similar to the size of the heated region, ~ 50 km in diameter. The fields are an order of magnitude lower in a larger spot of ~ 100 – 200 km diameter, and decrease very fast at larger distances from the center. The calculated fields are larger than $E_{\text{max}} \simeq 2$ $\mu\text{V m}^{-1}$ and $B_{\text{max}} \simeq 0.5$ pT measured by the Akebono satellite above the Tromsø heating facility in Norway [Kimura *et al.*, 1994], which is probably due to (1) a lower heating power than HAARP; and (2) the fact that the satellite was outside the narrow region where the maximum field is achieved.

Recent observations under similar conditions (HF frequency of 3.25 MHz and modulation at 2 kHz) by the DEMETER satellite indicate values as high as $E_{\text{max}} \simeq 350$ $\mu\text{V m}^{-1}$ and $B_{\text{max}} \simeq 20$ pT in a narrow “funnel” corresponding to the size (~ 10 km) of the radiating currents region [Pidtyachiy *et al.*, 2007]. However, these measurements were made for the upgraded HAARP facility, with a higher effective radiated power (ERP) of ~ 350 MW and a narrower beam (which explains a smaller radiating region). The total energy emitted upward is calculated by integrating the vertical Poynting flux and is equal to $\simeq 3$ W for the considered example, which is inside the range 0.32–4 W previously estimated from observations by DEMETER (before the upgrade) [Platino *et al.*, 2006].

3.2. Comparison with ground-based observations

The calculated ground-based horizontal magnetic field $\mathbf{B}_{\perp} = (\mathbf{H}_{\perp})_1/c$ and vertical electric field $(E_z)_1$ are shown in Figure 5. The magnetic field $\mathbf{B}_{\perp} \simeq 1$ – 2 pT is similar to the field measured at VLF sites in immediate vicinity of HAARP heating facility [Payne *et al.*, 2007]. The fields show the concentric structure of ELF waves propagating in the Earth-ionosphere waveguide. The ELF mode structure at all altitudes is shown in Figure 2b. The total emission power into the Earth-ionosphere waveguide can be calculated by integrating the horizontal Poynting flux through the boundary of a box enclosing the source currents, and is estimated to be $\simeq 1$ W (at the distance of 50 km from the central axis), which is slightly below the range 2.71–4.22 W obtained by Platino *et al.* [2006].

4. Discussion

The proposed method uses the idea due to Nygrén [1982] for recursive calculation of both the reflection coefficients and the wave amplitudes, the direction of recursion being important for the stability against “swamping”. For example, for upward-propagating waves the recursion is $\mathbf{R}_{k+1}^u \rightarrow \mathbf{R}_k^u$ and $\mathbf{u}_k \rightarrow \mathbf{u}_{k+1}$. This first relation can be understood as insertion of an additional slab of a refracting medium in the path of an upward propagating wave, which modifies the old reflection coefficient. The second relation is just following the direction of the wave propagation. Thus, the direction of the recursive calculation has a simple causal physical meaning, which probably explains the stability of the method against “swamping”.

The calculations in the heating example are performed assuming a stratified medium, i.e., we neglect the change of conductivity in a small region caused by heating, and consider the modulated electrojet currents as an external source radiating in an unmodified ionosphere. The maximum change in conductivity occurs at 80 km and is equal to ~ 80 nSm $^{-1}$ (for both Pedersen and Hall components) [Payne *et al.*, 2007]. This equals to relative changes of 22.5% for Pedersen and 3.5% for Hall components of the background conductivity. Since we neglect the effect of this change on the propagation of the electromagnetic waves in our calculations, we expect relative errors of the same order in the calculated fields.

5. Conclusions

We have developed a stable method for calculation of waves in a stratified medium which are created by an arbitrary combination of harmonically varying sources, with arbitrary electromagnetic properties of the medium (i.e., permittivity tensor $\hat{\epsilon}$). Currently, the only drawback of the application of this method to the generation of electromagnetic waves in the Earth-ionosphere waveguide and ionosphere by ionospheric heaters is that the curvature of the Earth is not taken into account. However, the method presented herein can be modified by including small corrections for the curvature.

Appendix A: Boundary conditions when crossing a double layer

Consider the field generated in vacuum by a current and charge density given by

$$\mathbf{J}(\mathbf{r}, t) = \hat{z} J \delta(z) e^{ik_x x - i\omega t} \quad (\text{A1})$$

$$\rho(\mathbf{r}, t) = \frac{J \delta'(z)}{i\omega} e^{ik_x x - i\omega t} \quad (\text{A2})$$

where ρ has been found from \mathbf{J} using the continuity equation $\nabla \cdot \mathbf{J} + \dot{\rho} = 0$. Potentials conforming with the Lorenz gauge condition $\nabla \cdot \mathbf{A} + \varepsilon_0 \mu_0 \dot{\phi} = 0$ satisfy equations

$$\nabla^2 \mathbf{A} - \varepsilon_0 \mu_0 \ddot{\mathbf{A}} = -\mu_0 \mathbf{J} \quad (\text{A3})$$

$$\nabla^2 \phi - \varepsilon_0 \mu_0 \ddot{\phi} = -\rho/\varepsilon_0 \quad (\text{A4})$$

For the sources given by (A1–A2), we look for a solution in the form $\mathbf{A}(\mathbf{r}, t) = \hat{z} A_z(z) e^{ik_x x - i\omega t}$ and $\phi(\mathbf{r}, t) = \phi(z) e^{ik_x x - i\omega t}$. Equation (A4) becomes:

$$k_z^2 \phi(z) + \partial_{zz} \phi(z) = -\frac{J \delta'(z)}{i\omega \varepsilon_0} \quad (\text{A5})$$

where $k_z = \sqrt{k_0^2 - k_x^2}$ and $k_0 = \omega/c$. Solving for $\phi(z)$ and obtaining $A_z(z)$ from the gauge condition, we find

$$\phi(\mathbf{r}, t) = -\frac{J \text{sign}(z)}{2i\omega \varepsilon_0} e^{ik_z |z| + ik_x x - i\omega t} \quad (\text{A6})$$

$$\mathbf{A}(\mathbf{r}, t) = -\hat{z} \frac{J \mu_0}{2ik_z} e^{ik_z |z| + ik_x x - i\omega t} \quad (\text{A7})$$

The fields are found from $\mathbf{E}(\mathbf{r}, t) = -\nabla \phi - \dot{\mathbf{A}}$, $\mathbf{H}(\mathbf{r}, t) = c \nabla \times \mathbf{A}$ (we use $\mathbf{H} \equiv Z_0 \mathbf{H}_{\text{SI}}$, as in the main text of this paper). Substituting expressions for ϕ and \mathbf{A} , we find

$$\mathbf{E}(\mathbf{r}, t) = \frac{Z_0 J}{2} \left(\hat{x} n_x \text{sign}(z) - \hat{z} \frac{n_x^2}{n_z} \right) e^{ik_z |z| + ik_x x - i\omega t} + \hat{z} \frac{Z_0 J}{ik_0} \delta(z) e^{ik_x x - i\omega t} \quad (\text{A8})$$

$$\mathbf{H}(\mathbf{r}, t) = \hat{y} \frac{Z_0 J n_x}{2 n_z} e^{ik_z |z| + ik_x x - i\omega t} \quad (\text{A9})$$

where $n_x = k_x/k_0$, $n_z = k_z/k_0$ and $Z_0 = \sqrt{\mu_0/\varepsilon_0}$. The change in the perpendicular component of \mathbf{E} is

$$\Delta \mathbf{E}_{\perp} = \mathbf{E}_{\perp}|_{z=+0} - \mathbf{E}_{\perp}|_{z=-0} = Z_0 J \mathbf{n}_{\perp} \quad (\text{A10})$$

where $\mathbf{n}_{\perp} = \hat{x} n_x$, and we omitted the factor $e^{ik_x x - i\omega t}$. We also see that $\Delta \mathbf{H}_{\perp} = 0$. In the configuration space, for

$\mathbf{J}(\mathbf{r}, t) = \hat{z}J(\mathbf{r}_\perp)\delta(z)e^{-i\omega t}$, the last relation becomes

$$\Delta \mathbf{E}_\perp(\mathbf{r}_\perp) = \frac{Z_0}{ik_0} \nabla_\perp J(\mathbf{r}_\perp) e^{-i\omega t} \quad (\text{A11})$$

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