(Everything a physicist needs to know about) 
Bessel functions $J_n(x)$ of integer order 
(and also Hankel functions $H_{n}^{(1,2)}$)

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Abstract 
Some properties of integer-order Bessel functions $J_n(x)$ are derived 
from their definition using the generating function. The results may be 
of use in such areas as plasma physics. Now with a Section on Hankel 
functions $H_{n}^{(1,2)}(x)$! We assume that the reader knows some complex 
analysis (e.g., can integrate in the complex plane using residues).

1 Basic properties

1.1 Generating function

We derive everything else from here, which will serve us the definition of the 
integer-order Bessel functions (of the first kind):

\[ g(x, t) = e^{\frac{x}{2}(t - \frac{1}{t})} = \sum_{n=-\infty}^{+\infty} J_n(x)t^n \]  

(1)

We immediately make our first observations that since $g(x, t)$ is real for real 
$x, t$ then $J_n(x)$ must be real for real $x$. From now on,

\[ \sum_n \text{ or just } \sum \quad \text{will mean} \quad \sum_{n=-\infty}^{+\infty} \]

\[ \int \frac{d\theta}{2\pi} \quad \text{will mean} \quad \int_{0}^{2\pi} \frac{d\theta}{2\pi} \]
1.2 Some useful formulas

1. \( t \to -t \) in (1) \[ e^{-\frac{x}{2}(t-\frac{1}{t})} = \sum (-1)^n J_n(x) t^n = \sum J_n(-x) t^n \implies J_n(-x) = (-1)^n J_n(x) \]

2. \( t \to \frac{1}{t} \) in (1) \[ e^{-\frac{x}{2}(t-\frac{1}{t})} = \sum J_n(-x) t^n = \sum J_n(x) t^{-n} \implies J_n(x) = (-1)^n J_n(x) = J_n(-x) \] (2)

3. \( t = e^{i\theta} \) \[ \frac{1}{2} (t - \frac{1}{t}) = i \sin \theta \implies e^{ix \sin \theta} = \sum J_n(x) e^{i n \theta} \] (3)

4. \[ \int \frac{d\theta}{2\pi} e^{-i m \theta} \times (3), \text{ switch } m \to n, \implies J_n(x) = \int \frac{d\theta}{2\pi} e^{i x \sin \theta - i n \theta} \] (4)

1.3 Recursion formulas

1. \[ \frac{\partial}{\partial x} (1) \implies \frac{1}{2} (t - \frac{1}{t}) g(x, t) = \sum J'_n(x) t^n \implies \sum \frac{1}{2} J_{n-1}(x) - J_{n+1}(x) = \sum J'_n(x) t^n \implies J'_n(x) = \frac{1}{2} (J_{n-1}(x) - J_{n+1}(x)) \] (5)

2. \[ \frac{\partial}{\partial t} (1) \implies \frac{x}{2} (1 + \frac{1}{t^2}) g(x, t) = \sum n J_n(x) t^{n-1}, \text{ multiply by } t, \implies \frac{x}{2} \left( \frac{t}{2} \right) g(x, t) = \frac{x}{2} \sum J_n(x) (t^{n-1} + t^{n+1}) = \sum n J_n(x) t^n \implies n J_n(x) = \frac{x}{2} (J_{n-1}(x) + J_{n+1}(x)) \] (6)

3. Combining the two we can get

\[ J'_n = -\frac{n}{x} J_n - J_{n+1} \] (7)

\[ J'_n = J_{n-1} - \frac{n}{x} J_n \] (8)
1.4 Differential equation

Bessel equation arises when we solve Helmholtz equation $\nabla^2 \phi + p^2 \phi = 0$ in 2D, in cylindrical coordinates. It is more naturally understood when we go to Fourier coordinates $k$ (see Section 3.1 below), in which operator $\nabla = ik$, $\nabla^2 = -k^2$. Here we brutally derive the Bessel equation from the recursion formulas.

Let us use (7) to find $J'_n$, differentiate it again and substitute expression $J'_{n+1} = J_n - \frac{n+1}{x} J_{n+1}$ derived from (8). We have

\[
\begin{align*}
J'_n &= \frac{n}{x} J_n - J_{n+1} \\
J''_n &= \left( \frac{n}{x} J_n \right)' - J_n + \frac{n+1}{x} J_{n+1} \\
&= -\frac{n}{x^2} J_n + \frac{n^2}{x^2} J_n - \frac{n}{x} J_{n+1} - J_n + \frac{n+1}{x} J_{n+1} \\
&= \frac{1}{x} \left( -\frac{n}{x} J_n + \frac{n^2}{x} J_n - x J_n + J_{n+1} \right)
\end{align*}
\]

By eliminating $J_{n+1}$ from these equations we obtain

\[x^2 J''_n + x J'_n + (x^2 - n^2) J_n = 0 \tag{9}\]

2 Sums

2.1 Sums of the first power of $J_n(x)$ or $J'_n(x)$

The basic idea is to substitute $t = 1$ in various derivatives of $g(x,t)$ in respect to both $x$ and $t$.

1. $t = 1$ in (1) $\implies \sum J_n(x) = 1 \tag{10}$

2. $\sum n \implies \sum n J_n(x) = x \tag{11}$

3. $\sum n^k J_n(x)$ may be found by differentiating $g(x,t)$ over $t$ and multiplying by $t$, $k$ times. Example:

$\sum n^2 J_n(x) t^n = t \frac{\partial}{\partial t} \left( t + \frac{1}{t} \right) g(x,t) = \ldots$
2.2 Sums of the second power of $J_n(x)$ or $J'_n(x)$

The basic idea is to multiply two $g(x, t)$ or its derivatives in respect to both $x$ and $t$. One can take the second $g$ to be a function of $u = 1/t$ instead. Note that for equal first arguments $g(x, t)g(x, u) = 1$ and we have a polynomial in $t$. By equating powers, we get various sums. Here are some examples.

1. $g(x, t)g(x, 1/t) = 1 = \sum_n \sum_m J_n(x)J_m(x)t^{n-m} \implies \text{all coefficients are zero except at } t^0 \implies \sum_n J_nJ_{n-k} = \delta_{k,0}$

2. $g(x, t)g(y, t) = \exp\left(\frac{x+y}{2} [t - 1/t]\right) \implies \sum_k J_k(x+y)t^k = \sum_n \sum_m J_n(x)J_m(y)t^{n+m} \implies$

$$J_k(x+y) = \sum_n J_n(x)J_{n-k}(y) = \sum_n J_n(x)J_{n-k}(-y)$$

where we used (2).

2.3 Graf’s addition theorem

See [Abramowitz and Stegun (1965), eq 8.1.79].

From Figure 1, $OB = z \sin(\theta + \psi) = OA + AB = x \sin \theta + y \sin(\phi - \theta)$.

From (4), $J_n(z) = \int \frac{d\theta}{2\pi} e^{iz \sin(\theta + \psi) - in(\theta + \psi)} \implies e^{in\psi} J_n(z) = \int \frac{d\theta}{2\pi} e^{ix \sin \theta + iy \sin(\phi - \theta) - in\theta}$

Substitute $e^{ix \sin \theta} = \sum_k J_k(x)e^{ik\theta}$, $e^{iy \sin(\phi - \theta)} = \sum_m J_m(y)e^{im(\phi - \theta)} \implies$
\[ e^{in\psi} J_n(z) = \sum_k \sum_m J_k(x) J_m(y) e^{im\phi} \int \frac{d\theta}{2\pi} e^{-i\theta + ik\theta - im\theta} \]

Use \[ \int \frac{d\theta}{2\pi} e^{i\theta} = \delta_{l,0} \] for integer \( l \) ⇒

\[ e^{in\psi} J_n(z) = \sum_m J_{m+n}(x) J_m(y) e^{im\phi} \] (12)

In particular, for \( n = 0 \)

\[ J_0(z) = J_0(\sqrt{x^2 + y^2 - 2xy \cos \phi}) = \sum_m J_m(x) J_m(y) e^{im\phi} \] (13)

3 Integrals

3.1 2D Fourier transforms

Define direct and inverse Fourier transform of a function \( F(r) \) in 2D position space \( r = \{x, y\} \) into function \( \tilde{F}(k) \) in 2D wavevector space \( k = \{k_x, k_y\} \) as

\[ \tilde{F}(k) = \int d^2 r e^{-ikr} F(r) \]

\[ F(r) = \int \frac{d^2 k}{(2\pi)^2} e^{ikr} \tilde{F}(k) \]

where \( d^2 r = dx \, dy \) and \( d^2 k = dk_x \, dk_y \) and the integration is over the whole plane. Below, we also use polar coordinates \( r = |r|, \theta = \text{atan}2(y, x) \) in position space and \( k = |k|, \chi = \text{atan}2(k_y, k_x) \) in wavevector space.

Let \( F(r) = \delta(r - a) e^{-in\theta} \). The Fourier image is

\[ \tilde{F}(k) = \int d^2 r e^{-ikr} F(r) = \int_0^\infty r \, dr \int_0^{2\pi} d\theta \, \delta(r - a) e^{-in\theta - ikr \cos(\theta - \chi)} \]

Introducing an auxiliary angle \( \phi = \theta - \chi - \pi/2 \), we get \( \cos(\theta - \chi) = -\sin \phi \) and \( \theta = \phi + \chi + \pi/2 \) ⇒

\[ \tilde{F}(k) = 2\pi a e^{-in\chi - in\pi/2} \int \frac{d\phi}{2\pi} e^{-in\phi + ika \sin \phi} = 2\pi a i^{-n} e^{-i\chi} J_n(ka) \]

We can use the same formula by substituting \( r \leftrightarrow k \) and \( i \leftrightarrow -i \) and
taking care of coefficient $(2\pi)^2$. It summary,

\[
F(r) \iff \tilde{F}(k)
\]

\[
\frac{\delta(r-a)}{a}e^{in\theta} \iff 2\pi i^{2-n} J_n(ka)e^{inx}
\]

(14)

\[
i^n J_n(pr)e^{in\theta} \iff \frac{2\pi \delta(k-p)}{p}e^{inx}
\]

(15)

3.2 Weber’s First Integral

See [Abramowitz and Stegun](1965), eq 11.4.28 with \( \mu = 2, \nu = 0 \).

The integral \( I_1 = \int_{0}^{\infty} r \, dr \, J_0(pr)e^{-q^2 r^2} \) is found by representing this integral in a 2D plane \( (2\pi r \, dr = d^2 r) \) and going to Fourier space \( r \to k \).

\[
I_1 = \int d^2 r \, F_1(r)F_2(r), \quad F_1(r) = e^{-q^2 r^2}, \quad F_2(r) = \frac{1}{2\pi} J_0(pr), \quad \text{where} \quad r = |r|.
\]

We use the fact that \( \int d^2 r \, F_1(r)F_2(r) = \int \frac{d^2 k}{(2\pi)^2} \tilde{F}_1(k)\tilde{F}_2(k) \), where \( \tilde{F}_{1,2}(k) = \int d^2 r \, e^{-ik \cdot r}F_{1,2}(r) \) are the Fourier images. We have \( \tilde{F}_1(k) = \frac{\pi}{q^2} e^{-\frac{k^2}{4q^2}} \) and using the Fourier transform result from above, we obtain \( \tilde{F}_2(k) = \frac{\delta(k-p)}{p} \).

Thus,

\[
\int_{0}^{\infty} x \, dx \, J_0(px)e^{-q^2 x^2} = \frac{1}{2q^2} e^{-\frac{p^2}{4q^2}}
\]

(16)

3.3 Weber’s Second Integral

\[
I_2 = \int_{0}^{\infty} x \, dx \, J_n(ax)J_n(bx)e^{-q^2 x^2}
\]

From Graf’s addition theorem (13) \implies \quad J_n(x)J_n(y) = \int \frac{d\phi}{2\pi} J_0(z)e^{-in\phi} \implies

\[
J_n(ax)J_n(bx) = \int \frac{d\phi}{2\pi} J_0(x\sqrt{a^2+b^2-2ab\cos\theta})e^{-in\phi} \implies
I_2 = \int \frac{d\phi}{2\pi} e^{-in\theta} \int_{0}^{\infty} x \, dx \, J_0(x\sqrt{a^2+b^2-2ab\cos\theta})e^{-q^2 x^2} \implies [\text{use } (16)] \implies
\]

\[
I_2 = \int \frac{d\phi}{2\pi} \frac{1}{2q^2} \exp\left[-\frac{a^2+b^2-2ab\cos\theta}{4q^2} - in\theta\right] = \frac{1}{2q^2} e^{-\frac{a^2+b^2}{4q^2}} \int \frac{d\phi}{2\pi} \exp\left[i(-\frac{ab}{2q^2})\sin\phi - in\phi + in\frac{\pi}{2}\right]
\]

(we substituted \( \theta = \phi - \pi/2 \))

\[
I_2 = \frac{1}{2q^2} e^{-\frac{a^2+b^2}{4q^2}} J_n\left(-i\frac{ab}{2q^2}\right)i^n
\]

We use the definition of the modified Bessel function (of the first kind)

\[
I_n(x) = i^n J_n(-ix) = i^{-n} J_n(ix)
\]
\[
\int_0^\infty x\,dx\ J_n(ax)J_n(bx)e^{-q^2x^2} = \frac{1}{2q^2}e^{-\frac{a^2+b^2}{4q^2}}I_n\left(\frac{ab}{2q^2}\right)
\]

(17)

In particular, for \(a = b\) we have

\[
\int_0^\infty x\,dx\ J_n^2(ax)e^{-q^2x^2} = \frac{1}{2q^2}e^{-\frac{a^2}{2q^2}}I_n\left(\frac{a^2}{2q^2}\right)
\]

where we introduced another function \(\Lambda_n\) whose properties are explored in more detail below in Section 5.

### 3.4 Lipschitz Integral

\[
I = \int_0^\infty e^{-ax}J_0(bx)\,dx = \int \frac{db}{2\pi} \int_0^\infty e^{-ax + ibx\sin \theta} \,dx = \int \frac{db}{2\pi} \frac{1}{a - ib\sin \theta}
\]

Substitute \(z = e^{i\theta}\), integration is now around the unit circle in the complex \(z\)-plane:

\[
I = \frac{1}{2\pi} \oint \frac{dz}{az - 1} = -\frac{1}{2\pi i} \oint \frac{dz}{z^2 - \frac{a^2}{b^2}} = -\frac{1}{2\pi i} \oint \frac{dz}{(z - z_1)(z - z_2)},\text{ where } z_{1,2} = \left(\frac{a \pm \sqrt{a^2 + b^2}}{b}\right).
\]

Note that \(z_1\) is outside the unit circle, \(z_2\) is inside. We use the residue calculus and find

\[
I = -\frac{1}{b} \left(\frac{1}{z_2 - z_1}\right) = \frac{1}{\sqrt{a^2 + b^2}}.
\]

\[
\int_0^\infty e^{-ax}J_0(bx)\,dx = \frac{1}{\sqrt{a^2 + b^2}}
\]

(18)

### 3.5 1D Fourier Integral

Calculation is similar to the Lipschitz integral.

\[
\int_{-\infty}^{+\infty} e^{-iax}J_0(bx)\,dx = \int \frac{2\pi}{2\pi} e^{-ib\theta} \int_{-\infty}^{+\infty} e^{-iax + ibx\sin \theta} \,dx = \int_0^{2\pi} e^{-ib\theta} \delta(a - b\sin \theta) \,d\theta
\]

where we used the integral representation property of the delta function. Its argument may be transformed using:

\[
\delta(f(x)) = \sum_k \frac{1}{|f'(x_k)|} \delta(x - x_k)
\]

where \(x_k\) are the zeros of \(f(x)\). If \(|a| \geq |b|\), there are no zeros. Otherwise, there are two: \(\theta_1 = \arcsin(a/b), \theta_2 = \pi - \arcsin(a/b)\).
We have \( \delta(a - b \sin \theta) = \frac{1}{|b \cos \arcsin(a/b)|} [\delta(\theta - \theta_1) + \delta(\theta - \theta_2)] \) and finally

\[
\int_{-\infty}^{+\infty} e^{-iax} J_n(bx) \, dx = \frac{H(|b| - |a|)}{\sqrt{b^2 - a^2}} \left[ u^{-n} + (-u)^n \right]
\]

where

\[
u = e^{i \arcsin(a/b)} = \sqrt{1 - \left(\frac{a}{b}\right)^2} + i \left(\frac{a}{b}\right)
\]

and \( H \) is the Heaviside step function.

## 4 Asymptotics

### 4.1 Taylor series expansion

We will derive the expansion of \( J_n(x) \) in the vicinity of \( x = 0 \) using \([4]\). Expand

\[
e^{ix \sin \theta} = \sum_{m=0}^{\infty} \frac{x^m (i \sin \theta)^m}{m!} = \sum_{m=0}^{\infty} \frac{(e^{i\theta} - e^{-i\theta})^m}{m!} \left(\frac{x}{2}\right)^m
\]

We use the binomial expansion

\[
\frac{(e^{i\theta} - e^{-i\theta})^m}{m!} = \sum_{k=0}^{m} \frac{(-1)^k e^{i[(m-k)-k]\theta}}{(m-k)!k!}
\]

The integral over \( \theta \) “cuts out” only the exponents with the correct argument:

\[
J_n(x) = \int \frac{d\theta}{2\pi} e^{ix \sin \theta - in\theta} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \delta_{n,m-2k} \frac{(-1)^k}{(m-k)!k!} \left(\frac{x}{2}\right)^m
\]

Let us assume \( n \geq 0 \) (the formulas for \( n < 0 \) are obtained trivially using equation \([2]\)). After substitution \( m = n + 2k \) instead of double summation we have summation in \( k \) only:

\[
J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!} \left(\frac{x}{2}\right)^{n+2k}
\]  \qquad (19)
4.2 Asymptotics for large $x$

Again, use (4). The phase $\phi = x \sin \theta - n \theta$ changes fast for large $x$, therefore we should use the stationary phase integration method. The stationary phase is at $\theta = \pm \theta_0$ found from $\phi'(\theta) = x \cos \theta - n = 0 \implies \cos \theta_0 = n/x \ll 1$, we can use $\sin \theta_0 \approx 1$, $\theta_0 \approx \pi/2$ and $\phi(\theta_0) \approx x - n\pi/2$. The second derivative at $\theta_0$ is $\phi''(\theta_0) = -x \sin \theta_0 \approx -x$. The contribution at $-\theta_0$ is a complex conjugate. Thus, we extend integration over $\zeta = \theta - \theta_0$ to infinite limits and get

$$J_n(x) \approx \exp[i\phi(\theta_0)] \int_{-\infty}^{+\infty} \frac{d\zeta}{2\pi} \exp[i\phi''(\theta_0)\zeta^2/2] + \text{c.c.}$$

$$= \exp[i(x - n\pi/2)] \int_{-\infty}^{+\infty} \frac{d\zeta}{2\pi} \exp[-i x \zeta^2/2] + \text{c.c.}$$

Using gaussian integral $\int e^{-\zeta^2/(2\sigma^2)} d\zeta = \sqrt{2\pi\sigma}$ with $\sigma = \sqrt{1/(ix)}$, we find

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{n\pi}{2} - \frac{\pi}{4} \right), \quad x \to \infty$$

and for modified functions,

$$I_n(x) \approx \frac{e^x}{\sqrt{2\pi x}}, \quad x \to \infty$$

5 Properties of $\Lambda_n(x)$

Introduce $\Lambda_n(x) = e^{-x}I_n(x)$. First, some properties of $I_n(x) = i^n J_n(-ix) = i^{-n} J_n(ix)$ which are derived elementarily from the corresponding properties of $J_n$:

$$x^2 I_n'' = -x I_n' + (x^2 + n^2)I_n$$

$$I_n' = \frac{1}{2}(I_{n-1} + I_{n+1})$$

$$nI_n = \frac{x}{2}(I_{n-1} - I_{n+1})$$

$$I_n - I_n = I_n$$

$$\sum_n I_n(x)e^{in\theta} = e^{x \cos \theta}$$
From here, it follows that $\Lambda'_n = e^{-x}(I'_n - I_n)$. Some important properties of $\Lambda_n$:

$$\sum_n \Lambda_n(x) = 1 \quad (\text{follows from the sum of } I_n \text{ above with } \theta = 0)$$

$$\sum_n n\Lambda_n(x) = \frac{x}{2} \sum_n (\Lambda_{n-1} - \Lambda_{n+1}) = 0$$

### 5.1 Taylor series

Let us derive Taylor series expansion of $\Lambda_n$. We can assume $n \geq 0$ because otherwise $\Lambda_n = \Lambda_{-n}$. We start with (see equation [19])

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}, \quad I_n(x) = \sum_{l=0}^{\infty} \frac{1}{l! (n+l)!} \left( \frac{x}{2} \right)^{2l+n}$$

Multiplying, we get

$$e^{-x}I_n(x) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-2)^k}{k! l! (n+l)!} \left( \frac{x}{2} \right)^{2l+n+k} = \sum_{p=n}^{\infty} A_p \left( \frac{x}{2} \right)^p$$

The goal is to find a tractable expression for

$$A_p = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-2)^k}{k! l! m!} \frac{1}{m-l}$$

We use the polynomial expansion

$$(a + b + c)^p = \sum_{k,l,m=0}^{\infty} \frac{p!}{k! l! m!} a^k b^l c^m$$

Take $a = -2$, need to single out term $b^l c^m$ such that $m-l = n$. Take $b = e^{i\theta}$, $c = e^{-i\theta}$, we get $b^l c^m = e^{-in\theta}$ which may be singled out by integration over $\theta$ of this expression multiplied by $e^{in\theta}$:

$$A_p = \frac{1}{p!} \int_0^{2\pi} (-2 + e^{i\theta} + e^{-i\theta})^p e^{in\theta} \frac{d\theta}{2\pi}$$

10
Let us simplify using $-2 + e^{i\theta} + e^{-i\theta} = -(2 \sin \theta')^2$, with $\theta' = \theta/2$. Integration over $\theta'$ is from 0 to $\pi$ but may be extended to $[0, 2\pi]$ because of the symmetry of the expression under the integral. After dropping the dash from $\theta'$ and using $2 \sin \theta = -i(e^{i\theta} - e^{-i\theta})$:

$$A_p = \frac{1}{p!} \int_0^{2\pi} (e^{i\theta} - e^{-i\theta})^{2p} e^{2i\theta} \frac{d\theta}{2\pi}$$

Use the binomial expansion:

$$(e^{i\theta} - e^{-i\theta})^{2p} = \sum_{j=0}^{2p} \frac{(2p)!}{j!(2p-j)!} (-1)^{2p-j} e^{i\theta(j-2p+j)}$$

From this expansion, the integration over $\theta$ only extracts the term $\propto e^{-2i\theta}$, i.e., $j = p - n$. Finally,

$$A_p = \frac{(2p)!}{p!(p-n)!(p+n)!} (-1)^{p+n}$$

and we have our final answer

$$\Lambda_n(x) = \sum_{k=n}^{\infty} \frac{(2k)!}{k!(k-n)!(k+n)!} (-1)^{n+k} \left(\frac{x}{2}\right)^k$$

(21)

### 5.2 Large $x$

We use (20) to find

$$\Lambda_n(x) = i^n J_n(-ix)e^{-x} \approx \sqrt{\frac{1}{2\pi x}}, \quad x \to \infty$$

### 6 Hankel functions

This is extra material which is used in completely different situations. In plasma physics, $J_n$ is usually used to describe the field of a plane wave seen by an oscillating particle, using the expansion [3]. On the other hand, Hankel functions $H_n^{(1,2)}(x)$ (also called first and second kind, respectively) are usually used to describe waves in a cylindrical system (to which $J_n$ may also have a limited applicability).
6.1 Cylindrical waves

We will heavily use notations from Subsection 3.1. Let us consider the Helmholtz (wave) equation in 2D. From equation (15) we see that the function \( \Phi(r) = i^n J_n(pr)e^{in\theta} \) in configuration \( r \)-space becomes \( \tilde{\Phi}(k) = \frac{2\pi}{p} \delta(k - p)e^{in\chi} \) in the wave vector \( k \)-space which satisfies the uniform Helmholtz equation \((-k^2 + p^2)\tilde{\Phi}(k) = 0\) or \((\nabla^2 + p^2)\Phi(r) = 0\). Notice that another way to show this is that by separating variables \( \theta \) and \( r \) we get the differential equation (9). Let us consider a Helmholtz equation with sources:

\[(\nabla^2 + p^2)\Phi(r) = -S(r)\]

or, in \( k \)-domain,

\[(-k^2 + p^2)\tilde{\Phi}(k) = -\tilde{S}(k)\]

(the minus sign is for convenience).

The simplest source would be a \( \delta \)-function in the origin, however, it is cylindrically symmetric and can only generate the azimuthal harmonic with \( n = 0 \). As a more general source, let us take, for example, a thin (\( \delta \)-function) source at distance \( a \) from origin, such as given by equation (14) (we switch \( n \rightarrow -n \) for later convenience), with Fourier image

\[\tilde{S}(k) = 2\pi i^{-n} J_n(ka)e^{-in\chi} = s(k)i^{-n}e^{-in\chi}\]

The solution in \( r \)-domain is

\[\Phi(r) = i^{-n} \int \frac{d^2k}{(2\pi)^2} \frac{s(k)e^{ik\cdot r - in\chi}}{k^2 - p^2}\]

The azimuthal dependence of the answer will be \( \Phi(r) = \Phi(\hat{y})e^{-in\theta} \), so it is sufficient to find \( \Phi \) at, say, \( r = r\hat{y} \) with \( r > 0 \), i.e. \( \Phi(r) = \Phi(r\hat{y})e^{in(\pi/2)} = i^n\Phi(r\hat{y}) \). Let us evaluate this integral in Cartesian coordinates \( k_x, k_y \):

\[\Phi(r) = \int \frac{d^2k}{(2\pi)^2} \frac{s(k)e^{ik\cdot r - in\chi}}{k^2 - p^2} = \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} F(k_x)\]

where \( F(k_x) \) is the result of integration over \( k_y \):

\[F(k_x) = \int_{-\infty}^{+\infty} dk_y \frac{s(k)e^{ik_y r - in\chi}}{2\pi} \frac{1}{k_y^2 - (p^2 - k_x^2)}\]
where \( k = \sqrt{k_x^2 + k_y^2} \) and \( \chi = \arctan(2k_y/k_x) \).

For the ease of integration in complex \( k_y \) plane, we assume \( s(k) \) is an arbitrary function that is analytic in the upper half-plane \( (\Imag k_y > 0) \) and also not very big so that \( s(k)e^{ik_y r} \to 0 \) at \( \Imag k_y \to +\infty \). For example, we may take the above expression for a thin source with \( a \to 0 \) (a small source) so using (19) \( s(k) \approx 2\pi(n!2^n)^{-1}(ak)^n [1 + O(ak)] \propto k^n \). Note that for a finite-size source, \( s(k) = 2\pi J_n(ka) \), at \( k_y = iX, X \to +\infty \) asymptotically becomes, using \( k \approx k_y = iX \):

\[
s(k) \approx \sqrt{8\pi iXa} \cos \left( iXa - \frac{n\pi}{2} - \frac{\pi}{4} \right) \approx i^n \sqrt{\frac{2\pi}{Xa}} e^{Xa}
\]

so the product \( s(k)e^{ik_y r} \sim e^{X(a-r)} \to 0 \) only when \( r > a \) (makes sense because the free waves are outside the source).

### 6.2 Outward waves: \( H_n^{(1)}(x) \)

To get an outward propagating wave solution, assume that \( p \) has a small positive imaginary part, \( p \to p + i\Delta \) (i.e., the medium has a small absorption which makes the inward wave infinite at \( r \to \infty \) and therefore the outward wave is the only one that we get). This gives us the rule for going around the pole at \( k \approx p \). Since \( r > 0 \), we must close the integration contour in the upper plane, i.e., go counterclockwise around the pole at \( k_y = \sqrt{p^2 - k_x^2} \) if \( p > |k_x| \) and the pole at \( k_y = i\sqrt{k_x^2 - p^2} \) when \( p < |k_x| \). In both cases, we can write

\[
F(k_x) = \frac{i}{2} \frac{s(p)e^{ipr \sin \chi - i\pi n}}{p \sin \chi}
\]

where

\[
p \cos \chi = k_x
\]

\[
p \sin \chi = \begin{cases} \sqrt{p^2 - k_x^2} & \text{when } p > |k_x| \text{ or } |\cos \chi| < 1 \\ i\sqrt{k_x^2 - p^2} & \text{when } p < |k_x| \text{ or } |\cos \chi| > 1 \end{cases}
\]

i.e., \( p \sin \chi \) is the \( k_y \) value at the pole. Changing the integration variable \( dk_x = -p \sin \chi \, d\chi \) we finally get

\[
\Phi(r) = -s(p) \frac{i}{2} \int_C \frac{d\chi}{2\pi} e^{ipr \sin \chi - i\pi n}
\]
where the contour $C$ is such that $\cos \chi$ goes through the values $\{-\infty, -1, 0, 1, +\infty\}$ and $\sin \chi$ goes through the values $\{+i\infty, 0, 1, +i\infty\}$. This corresponds to a contour in $\chi$ going through points $\{\pi - i\infty, \pi, \pi/2, 0, +i\infty\}$. For convenience, usually a contour $C_1$ is chosen which goes in the opposite direction. We define (compare to (4))

$$H_n^{(1)}(x) = 2 \int_{C_1} \frac{d\chi}{2\pi} e^{ix\sin \chi - in\chi}, \quad C_1 = \{+i\infty, 0, \pi/2, \pi, \pi - i\infty\} \quad (22)$$

Thus

$$\Phi(r) = s(p) \frac{i}{4} H_n^{(1)}(pr) e^{-in\theta}$$

In particular, for a point source $S(r) = \delta(r)$ we have $n = 0$ and $s(p) = 1$ so the “outward” Green’s function of Helmholtz equation is

$$G(r) = \frac{i}{4} H_0^{(1)}(pr)$$

### 6.3 Inward waves: $H_n^{(2)}(x)$

If we look for inward waves, we set $p \to p - i\Delta$ and the pole in the upper half of $k_y$-plane is at $k_y = -\sqrt{p^2 - k_x^2}$ if $p > |k_x|$ and at $k_y = i\sqrt{k_x^2 - p^2}$ when $p < |k_x|$. We get the same expression for $F(k)$ but now

$$\begin{align*}
p \cos \chi &= k_x \\
\sin \chi &= \begin{cases} -\sqrt{p^2 - k_x^2} & \text{when } p > |k_x| \text{ or } |\cos \chi| < 1 \\
\frac{i}{\sqrt{k_x^2 - p^2}} & \text{when } p < |k_x| \text{ or } |\cos \chi| > 1 
\end{cases}
\end{align*}$$

Using $dk_x = -p \sin \chi d\chi$ we get

$$\Phi(r) = -s(p) \frac{i}{2} \int_{C_2} \frac{d\chi}{2\pi} e^{ipr\sin \chi - in\chi}$$

where the contour $C_2$ is such that $\cos \chi$ goes through the values $\{-\infty, -1, 0, 1, +\infty\}$ and $\sin \chi$ goes through the values $\{+i\infty, 0, -1, 0, +i\infty\}$. This corresponds to a contour in $\chi$ going through points $\{\pi - i\infty, \pi, 3\pi/2, 2\pi, 2\pi + i\infty\}$. We define

$$H_n^{(2)}(x) = 2 \int_{C_2} \frac{d\chi}{2\pi} e^{ix\sin \chi - in\chi}, \quad C_2 = \{\pi - i\infty, \pi, 3\pi/2, 2\pi, 2\pi + i\infty\} \quad (23)$$
and

$$\Phi(r) = -s(p)\frac{i}{4}H_n^{(2)}(pr)e^{-in\theta}$$

The “inward” Green’s function is

$$G(r) = -\frac{i}{4}H_0^{(2)}(pr)$$

Because of periodicity in $\chi$ with period $2\pi$, we can use $C_2 = \{-\pi-i\infty, -\pi, 0, +i\infty\}$ (for integer $n$, of course).

### 6.4 General properties

$H_n^{(1,2)}(x)$ satisfy the same recursion relations (5–6) and the differential equation (9) as $J_n(x)$ because they are defined through an integral similar to (4), although the contours of integration are different. Let us consider the symmetry relations analogous to equation (2) in more detail. In (22) for $-n$, change variable to $\alpha = \pi - \theta$ ($\theta = \pi - \alpha$). The contour $C_1$ changes to $-C_1$ (by minus we mean oppositely directed):

$$H_{-n}^{(1)}(x) = 2 \int_{\alpha \in -C_1} \frac{d(-\alpha)}{2\pi} e^{i\alpha \sin(\pi - \alpha)+in\pi - in\alpha} = e^{in\pi} H_n^{(1)}(x)$$

which is also valid for non-integer $n$ if we postulate the definition (22). For integer $n$, $e^{in\pi} = (-1)^n$.

Changing the variable to $\alpha = -\theta^*$, and using $\sin \alpha^* = (\sin \alpha)^*$, we get (the contour $C_1$ for $\theta$ becomes $-C_2$ for $\alpha$)

$$H_n^{(1)}(x) = 2 \int_{\alpha \in -C_2} \frac{d(-\alpha^*)}{2\pi} e^{-i\alpha \sin \alpha^* + in\alpha^*} = (H_n^{(2)}(x^*))^*$$

In summary,

$$H_n^{(1,2)}(x) = (-1)^n H_{-n}^{(1,2)}(x) = (H_n^{(2,1)}(x^*))^* \quad (24)$$

There is no relation of symmetry when changing $x \rightarrow -x$ because for $\Re x < 0$ the defining integral (22) becomes infinite.

We notice that $C_1 + C_2 = [0, 2\pi]$ (the parts going over complex $\theta$ cancel, also due to $2\pi$ periodicity). Thus,

$$J_n(x) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta \sin \theta - in\theta} = \frac{1}{2} (H_n^{(1)}(x) + H_n^{(2)}(x))$$
so \( J_n(x) = \Re H_n^{(1,2)}(x) \) for real \( x \). We may introduce Bessel functions of the second kind (Weber or Neumann functions) \( Y_n(x) \) which are real for real \( x \):

\[
\begin{align*}
H_n^{(1)}(x) &= J_n(x) + iY_n(x) \\
H_n^{(2)}(x) &= J_n(x) - iY_n(x)
\end{align*}
\]

But they are only useful as linearly independent from \( J_n(x) \) solutions of Helmholtz equation (or Bessel equation) and don’t have extra physical meaning compared to \( H_n^{(1,2)}(x) \). Moreover, \( Y_n(x) \) are infinite at \( x \to 0 \). This means there is no Taylor expansion at \( x \to 0 \).

One of the recursion formulas (5) is also valid for \( H_n^{(1,2)} \). The reason is that it may be derived from (4):

\[
\frac{d}{dx} Z_n(x) = \frac{d}{dx} \int_C \frac{d\theta}{2\pi} e^{ix\sin\theta-in\theta} = \int_C \frac{d\theta}{2\pi} i\sin\theta e^{ix\sin\theta-in\theta} = \frac{1}{2} \int_C \frac{d\theta}{2\pi} (e^{ix\sin\theta-in\theta+i\theta} - e^{ix\sin\theta-in\theta+i\theta}) = \frac{1}{2} (Z_{n-1}(x) - Z_{n+1}(x))
\]

where the contour of integration \( C \) is different for different Bessel functions \( Z \): (i) for \( Z \equiv J, C = \{\chi, \chi + 2\pi\} \) with arbitrary shift \( \chi \), due to periodicity of the expression under the integral; and (ii) for \( Z \equiv H^{(1,2)}, C = C_{1,2} \). The other recursion formula (6) may be derived for \( J \) by differentiating in the arbitrary phase shift \( \chi \), and is therefore not applicable to \( H^{(1,2)} \).

### 6.5 Asymptotics

We noted that there is no Taylor expansion of \( H_n^{(1,2)} \) at \( x \to 0 \). However, we may try to find the asymptotic expansion or at least the biggest term. Because of the symmetries, it is sufficient to consider only \( H_n^{(1)} \) and \( n \geq 0 \). For \( n = 0 \) we may use the Green’s function at \( pr \to 0 \), i.e., take \( p = 0 \) in the Helmholtz equation and compare to the Poisson equation Green’s function

\[
G_{\text{Poisson}}(r) = -\frac{\log r}{2\pi} + \text{const} = -\frac{\log(pr/2)}{2\pi} + \text{const} \approx \frac{i}{4} H_0^{(1)}(pr)
\]

from where \( H_0^{(1)}(x) \approx i(2/\pi) \log(x/2) + \text{const} \). We may neglect the constant at very small \( x \).

For \( n \geq 1 \), write out (22) explicitly, while changing the integration variable so that \( \theta = it \) on the interval \( \theta \in [i\infty, 0] \) and \( \theta = \pi - it \) on the interval
\[ \theta \in [\pi, \pi - i\infty] : \]

\[ H_n^{(1)}(x) = -\frac{i}{\pi} \int_0^\infty e^{-x \sinh t + nt} \, dt + \frac{1}{\pi} \int_0^\pi e^{ix \sin \theta - in\theta} \, d\theta - \frac{i(-1)^n}{\pi} \int_0^\infty e^{-x \sinh t - nt} \, dt \]

The middle term is finite for \( x = 0 \) and thus does not contribute to the infinite value. Regarding the last term we notice that for \( x, t > 0 \) we have \( \sinh t > t, \) \( e^{-x \sinh t - nt} < e^{-(x+n)t} \) and therefore

\[ \int_0^\infty e^{-x \sinh t - nt} \, dt < \frac{1}{x + n} < \frac{1}{n} \text{ (finite)} \]

In the first term, change the variable to \( u = x \sinh t \) or \( t = \log \left( \frac{u + \sqrt{u^2 + x^2}}{x} \right) : \)

\[ \int_0^\infty e^{-x \sinh t + nt} \, dt = \frac{1}{x^n} \int_0^\infty e^{-u(u + \sqrt{u^2 + x^2})^n} \, du \]

We may set \( x = 0 \) in the integral because it is finite:

\[ \int_0^\infty e^{-x \sinh t + nt} \, dt \approx \frac{1}{x^n} \int_0^\infty e^{-u(2u)^n} \, du = (n - 1)! \left( \frac{x}{2} \right)^{-n} \]

Collecting everything, we have for the biggest term

\[ H_n^{(1)}(x) \approx \begin{cases} 
- \frac{i}{\pi} \left( \frac{2}{\pi} \right) \log \left( \frac{2}{x} \right) + \text{const}, & n = 0 \\
- \frac{i(-1)^n}{\pi} \left( \frac{2}{x} \right)^n, & n > 0 
\end{cases} \] (25)

As expected, it is purely imaginary because the real part (\( = J_n(x) \)) is finite.

Asymptotics of \( H_n^{(1,2)}(x) \) at \( x \to \infty \) is obtained in the same way as we did for \( J_n(x) \) above in equation (20). This time, we have only one stationary phase point at \( \theta_0 = \arccos(n/x) \approx \pi/2 \) on contour \( C_1 \) (or \( \theta_0 = 2\pi - \arccos(n/x) \approx 3\pi/2 \) on contour \( C_2 \)). The value at large \( x \) is therefore

\[ H_n^{(1,2)}(x) \approx \sqrt{\frac{2}{\pi x}} \exp \left[ \pm i \left( x - \frac{n\pi}{2} - \frac{\pi}{4} \right) \right], \quad x \to \infty \] (26)

which makes it obvious that they represent outward and inward propagating waves. The non-oscillating contributions from intervals \( \theta \in [+i\infty, 0] \) and \( \theta \in [\pi, \pi - i\infty] \) contribute a smaller amount since \( e^{ix \sin \theta - in\theta} \approx e^{-x|\theta|} \) with integral (from both intervals) \( \approx 2/(\pi x) \).
6.6 Application to some integrals

Let us show that we can “straighten” our contours:

\[ C_1 \to C'_1 = \{+i\infty + \pi/2, -i\infty + \pi/2\} \]
\[ C_2 \to C'_2 = \{-i\infty + 3\pi/2, +i\infty + 3\pi/2\} \text{ or } \{-i\infty - \pi/2, +i\infty - \pi/2\} \]

Let us consider only \( C_1 \) here (\( C_2 \) is done analogously, or just use symmetry \([24]\)).

The difference in integration is \( C_1 - C'_1 = C_1^+ + C_1^- \), over two short contours with \( X \to +\infty \):

\[ C_1^+ = \{iX, iX + \pi/2\}, \quad C_1^- = \{-iX + \pi/2, -iX + \pi\} \]

The integrated function is analytic, so integration over a closed contour gives zero.

In the integral over \( C_1^+ \) we can change the variable to \( \theta = \chi - iX \) which is then integrated in \([0, \pi/2]\). The asymptotic behaviour of the exponent of the function under integral in \([22]\) on \( C_1^+ \) is

\[ ix \sin \chi - in\chi = ix \sin(\theta + iX) - in(\theta + iX) \approx -\frac{x}{2} e^{-x-i\theta} \]

The integral over \( C_1^+ \) will vanish only if the real part of the above expression becomes large and negative. This is equivalent to \( \arg x - \theta \in [-\pi/2, \pi/2] \), and \( C_1^+ \) is covered if \( \arg x \in [0, \pi/2] \).

Analogously, for \( C_1^- \), the exponent of the function under integral after change \( \theta = \chi + iX \) (where \( \theta \in [\pi/2, \pi] \)) behaves as

\[ ix \sin \chi - in\chi = ix \sin(\theta - iX) - in(\theta - iX) \approx \frac{x}{2} e^{-x+i\theta} \]

The integral over \( C_1^- \) will vanish if \( \arg x + \theta \in [\pi/2, 3\pi/2] \), and \( C_1^- \) is covered again if \( \arg x \in [0, \pi/2] \).

Thus, contributions from \( C_1^\pm \) vanish for \( \arg x \in [0, \pi/2] \) and integration over \( C_1 \) may be changed to integration over \( C_1' \). Analogously, \( C_2 \) may be changed to \( C_2' \) if \( \arg x \in [-\pi/2, 0] \). After change \( \chi = -i\xi + \pi/2 \), we get an alternative definition of \( H_n^{(1)} \):

\[ H_n^{(1)}(x) = \frac{(-i)^{n+1}}{\pi} \int_{-\infty}^{+\infty} e^{ix \cosh \xi - n\xi} d\xi \]
Using symmetry \([24]\), we arrive at the combined formula

\[
H_{n}^{(1,2)}(x) = \frac{(\mp i)^{n+1}}{\pi} \int_{-\infty}^{+\infty} e^{\pm ix \cosh \xi} \xi^{n} d\xi
\]

(27)

We see that these integrals actually converge in wider ranges \(\arg x \in [0, \pi]\) and \(\arg x \in [-\pi, 0]\), respectively, for which the original definitions \([22, 23]\) do not always converge. Using symmetry \([24]\), or by inverting the integration \((\xi \rightarrow -\xi)\) we may show that in both formulas we may switch \(+n\xi \leftrightarrow -n\xi\) in the exponent.

The above integrals may occur in a form such that \(u = \sinh \xi\). Then

\[
\begin{align*}
\cosh \xi &= \sqrt{u^2 + 1} \\
d\xi &= \frac{d\sinh \xi}{\cosh \xi} = \frac{du}{\sqrt{u^2 + 1}} \\
e^{\pm \xi} &= \cosh \xi \pm \sinh \xi = \sqrt{u^2 + 1} \pm u
\end{align*}
\]

and integration limits stay as \([-\infty, +\infty]\):

\[
H_{n}^{(1,2)}(x) = \frac{(\pm i)^{n+1}}{\pi} \int_{-\infty}^{+\infty} \frac{(u^2 + 1 + u)^n e^{\pm ix \sqrt{u^2 + 1}} du}{\sqrt{u^2 + 1}}
\]

(28)

where (a) the exponent \(n\) may be also changed to \(-n\) and/or (b) \(\sqrt{u^2 + 1} + u\) may be changed to \(\sqrt{u^2 + 1} - u\) in either formula.

These integrals may occur, in particular, in calculations of spacetime-domain propagators in relativistic quantum field theory (QFT) \([Greiner and Reinhardt 2009\) p. 68].

### 6.7 Modified Bessel functions of the second kind

When the argument is imaginary, we have the modified Bessel functions of the second kind (Basset or Macdonald functions):

\[
K_n(x) = \begin{cases} 
\frac{\pi}{2} n^{n+1} H_n^{(1)}(ix) & -\pi < \arg x \leq \frac{\pi}{2} \\
\frac{\pi}{2} (-i)^n H_n^{(2)}(-ix) & -\frac{\pi}{2} < \arg x \leq \pi 
\end{cases}
\]

(29)

From either of expressions \([27]\) for \(H^{(1,2)}\) we get:

\[
K_n(x) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-x \cosh \xi \pm n\xi} d\xi
\]
This formula confirms the symmetry \( K_{-n}(x) = K_n(x) \). From (28):

\[
K_n(x) = \int_{-\infty}^{+\infty} \frac{(\sqrt{u^2 + 1} \pm u)^n e^{-x\sqrt{u^2 + 1}}}{2\sqrt{u^2 + 1}} du
\]

where any choice of signs is valid.

### 6.7.1 Another integral representation

We can find another integral representation of \( K_n(x) \) by shifting the integration line in \( \xi \) by an imaginary number (this is equivalent to going back to the \( \chi \)-plane and shifting the contour \( C'_1 \) again to \( \{ +i\infty, -i\infty \} \)). Let us take \( \xi = \zeta - \frac{i\pi}{2} \). Then \( \cosh \xi = \cos(i\xi) = \cos(i\zeta + \pi/2) = -\sin(i\zeta) = -i\sinh \zeta \) so we have

\[
K_n(x) = \frac{1}{2} \int_{-\infty+i\frac{\pi}{2}}^{+\infty+i\frac{\pi}{2}} e^{ix \sinh \zeta \pm n\zeta} d\zeta
\]

We want to shift the integration contour back to the real axis. Let us see what is going on at the ends of this integral:

\[
2(\pm i)^n K_n(x) = \int_{-\infty}^{+\infty} e^{ix \sinh \zeta \pm n\zeta} d\zeta + \int_{-\infty+i\frac{\pi}{2}}^{-X} e^{ix \sinh \zeta \pm n\zeta} d\zeta + \int_{X+i\frac{\pi}{2}}^{+\infty} e^{ix \sinh \zeta \pm n\zeta} d\zeta
\]

We are interested only in asymptotic behavior \( X \to +\infty \). If the second and third integral vanish, we can do the aforementioned shift. Let us check if this is true.

In the second integral, change variable to \( \zeta = -X + i\theta \):

second integral \( \approx -i \int_{0}^{\frac{\pi}{2}} e^{(-xe^{X/2})[ie^{-i\theta}] + nX \pm in\theta} d\theta \)

In the third integral, change variable to \( \zeta = X + i\theta \):

third integral \( \approx i \int_{0}^{\frac{\pi}{2}} e^{(-xe^{X/2})[-ie^{i\theta}] + nX \pm in\theta} d\theta \)

We are in luck again and both of these integrals vanish for \( x > 0 \) and \( n = 0 \) because \( \Re\{ie^{-i\theta}\} > 0 \) and \( \Re\{-ie^{i\theta}\} > 0 \) on the interval \( \theta \in [0, \pi/2] \). So we get another integration rule

\[
K_0(x) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{ix \sinh \zeta} d\zeta \quad (30)
\]
which may be written as

\[ K_0(x) = \int_0^{+\infty} \cos(x \sinh \zeta) \, d\zeta \]

This formula is also used in QFT (Greiner and Reinhardt, 2009, p. 72), and in the last form, it is presented in Abramowitz and Stegun (1965, equation (9.6.21)). Note that unfortunately there is no “wiggle room” either in arg \( x \) (it has to be \( x > 0 \)) or \( n \) (it has to be \( n = 0 \)).

What happens when \( x > 0 \) but \( n \neq 0 \)? Consider the second integral, in which for concreteness choose the lower sign:

\[
2\text{nd integral} \approx -i \int_0^{\pi} e^{-(x/2)e^X} \sin \theta + nX - i(x/2)e^X \cos \theta + \theta | \, d\theta
\]

The biggest contribution will be for \( \theta \approx 0 \), so we can just write

\[
2\text{nd integral} \approx -ie^{nX-i(x/2)e^X} \int_0^{\infty} e^{-(x/2)e^X} \, d\theta = -ie^{nX-i(x/2)e^X} \frac{(x/2)e^X}{(x/2)e^X} = \frac{2}{i} e^{(n-1)X-i(x/2)e^X}
\]

Analogously, we find

\[
3\text{rd integral} \approx \frac{2i}{x} e^{-(n+1)X+i(x/2)e^X}
\]

and finally

\[
2(-i)^n K_n(x) = \int_{-\infty}^{+\infty} e^{ix \sinh \zeta - n\zeta} \, d\zeta + \frac{2}{ix} \left[ e^{(n-1)X-i(x/2)e^X} - e^{-(n+1)X+i(x/2)e^X} \right]
\]

The second term is infinite for \( |n| > 1 \) and finite, but rapidly oscillating, for \( n = \pm 1 \). Thus, the shift to the real axis is only strictly valid for \( n = 0 \).

Let us consider case \( n = 1 \) more closely. The above formula gives two equivalent presentations (taking \( n = \pm 1 \)):

\[
K_1(x) = \pm i \int_{-\infty}^{+\infty} e^{ix \sinh \zeta} \, d\zeta + \frac{e^{\mp i(x/2)e^X}}{x}
\]

In physical situations, we might be able to get rid of the oscillating term by shifting the contour into the complex domain just a very bit so that the exponent vanishes.
On the other hand, from recursion relation (5) (which is also valid for \( H^{(1,2)} \)), symmetry in \( n \) and definition (29), we have \( K'_0(x) = -K_1(x) \). Thus, we find

\[
K_1(x) = -\frac{1}{2} \frac{d}{dx} \int_{-\infty}^{+\infty} e^{ix\sinh \zeta} d\zeta = -\frac{1}{2} \int_{-\infty}^{+\infty} i \sinh \zeta e^{ix\sinh \zeta} d\zeta
\]
\[
= -\frac{i}{4} \left[ \int_{-\infty}^{+\infty} e^{ix\sinh \zeta} d\zeta + \int_{-\infty}^{+\infty} e^{ix\sinh \zeta} d\zeta \right]
\]

This is the average of the two versions of the above formula, with the neglect of the oscillating term.

6.7.2 Asymptotics

Asymptotics for \( K \) are obtained directly from its definition (29) and asymptotics of \( H^{(1,2)} \) (25,26):

\[
K_n(x) \approx \begin{cases} 
\log \left( \frac{2}{x} \right) + \text{const}, & n = 0 \\
\frac{(n-1)!}{2} \left( \frac{2}{x} \right)^n, & n > 0 
\end{cases} \quad x \to 0
\]

and

\[
K_n(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}, \quad x \to \infty
\]

References

Abramowitz, M., and I. A. Stegun (1965), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover.