(Everything a physicist needs to know about) Bessel functions $J_n(x)$ of integer order (and also Hankel functions $H_n^{(1,2)}$)

Nikolai G. Lehtinen

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Abstract

Some properties of integer-order Bessel functions $J_n(x)$ are derived from their definition using the generating function. The results may be of use in such areas as plasma physics. Now with a Section on Hankel functions $H_n^{(1,2)}(x)$! We assume that the reader knows some complex analysis (e.g., can integrate in the complex plane using residues).

1 Basic properties

1.1 Generating function

We derive everything else from here:

$$g(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{+\infty} J_n(x) t^n \tag{1}$$

We immediately make our first observations that since $g(x,t)$ is real for real x, t then $J_n(x)$ must be real for real x. From now on,

$$\sum_n \text{ or just } \sum \qquad \text{will mean} \qquad \sum_{n=-\infty}^{+\infty}$$

$$\int \frac{d\theta}{2\pi} \qquad \text{will mean} \qquad \int_0^{2\pi} \frac{d\theta}{2\pi}$$

1.2 Some useful formulas

1. $t \to -t$ in (1) $\implies$ $e^{-\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum (-1)^n J_n(x) t^n = \sum J_n(-x) t^n$ $\implies$ (equate powers of t) $\implies$

$$J_n(-x) = (-1)^n J_n(x)$$

2. $t \to \frac{1}{t}$ in (1) $\implies$ $e^{-\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum J_n(-x) t^n = \sum J_n(x) t^{-n}$ $\implies$ (equate powers of t and use previous expression) $\implies$

$$J_{-n}(x) = (-1)^n J_n(x) = J_n(-x) \tag{2}$$

3. $t = e^{i\theta}$ $\implies$ $\frac{1}{2}\left(t - \frac{1}{t}\right) = i \sin\theta$ $\implies$

$$e^{ix\sin\theta} = \sum J_n(x) e^{in\theta} \tag{3}$$

4. $\int \frac{d\theta}{2\pi} e^{-im\theta} \times (3)$, switch $m \to n$, $\implies$

$$J_n(x) = \int \frac{d\theta}{2\pi} e^{ix\sin\theta - in\theta} \tag{4}$$

1.3 Recursion formulas

1. $\frac{\partial}{\partial x}(1)$ $\implies$ $\frac{1}{2}\left(t - \frac{1}{t}\right) g(x,t) = \sum J_n'(x) t^n$ $\implies$
$\sum \frac{1}{2} J_n(x)(t^{n+1} - t^{n-1}) = \sum J_n'(x) t^n$ $\implies$
$\sum \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)) t^n = \sum J_n'(x) t^n$ $\implies$

$$J_n'(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)) \tag{5}$$

2. $\frac{\partial}{\partial t}(1)$ $\implies$ $\frac{x}{2}\left(1 + \frac{1}{t^2}\right) g(x,t) = \sum n J_n(x) t^{n-1}$, multiply by t, $\implies$
$\frac{x}{2}\left(t + \frac{1}{t}\right) g(x,t) = \frac{x}{2} \sum J_n(x)(t^{n-1} + t^{n+1}) = \sum n J_n(x) t^n$ $\implies$
$\frac{x}{2} \sum (J_{n+1}(x) + J_{n-1}(x)) t^n = \sum n J_n(x) t^n$ $\implies$

$$n J_n(x) = \frac{x}{2}(J_{n-1}(x) + J_{n+1}(x)) \tag{6}$$

3. Combining the two we can get

$$J_n' = \frac{n}{x} J_n - J_{n+1} \tag{7}$$

$$J_n' = J_{n-1} - \frac{n}{x} J_n \tag{8}$$

1.4 Differential equation

Bessel equation arises when we solve Helmholz equation $\nabla^2\phi + p^2\phi = 0$ in 2D, in cylindrical coordinates. It is more naturally understood when we go to Fourier coordinates $\mathbf{k}$ (see Section 3.1 below), in which operator $\nabla = i\mathbf{k}$, $\nabla^2 = -k^2$. Here we brutally derive the Bessel equation from the recursion formulas.

Let us use (7) to find J_n', differentiate it again and substitute expression $J_{n+1}' = J_n - \frac{n+1}{x}J_{n+1}$ derived from (8). We have

$$\left.\begin{array}{rcl} J_n' &=& \frac{n}{x}J_n - J_{n+1} \\ J_n'' &=& \left(\frac{n}{x}J_n\right)' - J_n + \frac{n+1}{x}J_{n+1} \\ &=& -\frac{n}{x^2}J_n + \frac{n^2}{x^2}J_n - \frac{n}{x}J_{n+1} - J_n + \frac{n+1}{x}J_{n+1} \\ &=& \frac{1}{x}(-\frac{n}{x}J_n + \frac{n^2}{x}J_n - xJ_n + J_{n+1}) \end{array}\right\}$$

By eliminating J_{n+1} from these equations we obtain

$$x^2J_n'' + xJ_n' + (x^2 - n^2)J_n = 0 \tag{9}$$

2 Sums

2.1 Sums of the first power of $J_n(x)$ or $J_n'(x)$

The basic idea is to substitute $t = 1$ in various derivatives of $g(x,t)$ in respect to both x and t.

1. $t = 1$ in (1) $\implies$
$$\sum J_n(x) = 1 \tag{10}$$
2. $\sum(6)$ $\implies$
$$\sum nJ_n(x) = x \tag{11}$$
3. $\sum n^kJ_n(x)$ may be found by differentiating $g(x,t)$ over t and multiplying by t, k times. Example:
$\sum n^2J_n(x)t^n = t\frac{\partial}{\partial t}\frac{x}{2}\left(t + \frac{1}{t}\right)g(x,t) = \ldots$

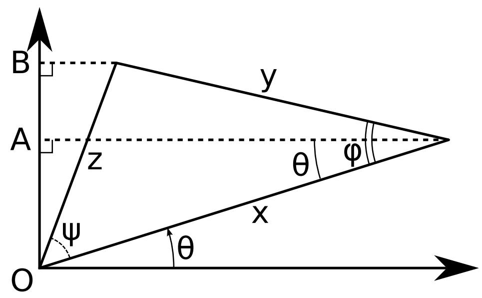

Figure 1: Graf's addition theorem

2.2 Sums of the second power of $J_n(x)$ or $J'_n(x)$

The basic idea is to multiply **two** $g(x,t)$ or its derivatives in respect to both x and t. One can take the second g to be a function of $u = 1/t$ instead. Note that for equal first arguments $g(x,t)g(x,u) = 1$ and we have a polynomial in t. By equating powers, we get various sums. Here are some examples.

1. $g(x,t)g(x,1/t) = 1 = \sum_n \sum_m J_n(x)J_m(x)t^{n-m} \implies$ all coefficients are zero except at $t^0 \implies$

$$\sum_n J_n J_{n-k} = \delta_{k,0}$$

2. $g(x,t)g(y,t) = \exp\left(\frac{x+y}{2}\left[t - 1/t\right]\right) \implies$
$\sum_k J_k(x+y)t^k = \sum_n \sum_m J_n(x)J_m(y)t^{n+m} \implies$

$$J_k(x+y) = \sum_n J_n(x)J_{k-n}(y) = \sum_n J_n(x)J_{n-k}(-y)$$

where we used (2).

2.3 Graf's addition theorem

See *Abramowitz and Stegun* (1965, eq 8.1.79).

From Figure 1, $OB = z\sin(\theta+\psi) = OA + AB = x\sin\theta + y\sin(\phi-\theta)$.
From (4), $J_n(z) = \int \frac{d\theta}{2\pi} e^{iz\sin(\theta+\psi)-in(\theta+\psi)} \implies e^{in\psi}J_n(z) = \int \frac{d\theta}{2\pi} e^{ix\sin\theta+iy\sin(\phi-\theta)-in\theta}$
Substitute $e^{ix\sin\theta} = \sum_k J_k(x)e^{ik\theta}$, $e^{iy\sin(\phi-\theta)} = \sum_m J_m(y)e^{im(\phi-\theta)} \implies$

$e^{in\psi}J_n(z) = \sum_k \sum_m J_k(x)J_m(y)e^{im\phi} \int \frac{d\theta}{2\pi}\, e^{-in\theta+ik\theta-im\theta}$
Use $\int \frac{d\theta}{2\pi}\, e^{il\theta} = \delta_{l,0}$ for integer $l \quad \Longrightarrow$

$$e^{in\psi}J_n(z) = \sum_m J_{m+n}(x)J_m(y)e^{im\phi} \tag{12}$$

In particular, for $n = 0$

$$J_0(z) = J_0(\sqrt{x^2+y^2-2xy\cos\phi}) = \sum_m J_m(x)J_m(y)e^{im\phi} \tag{13}$$

3 Integrals

3.1 Fourier transforms

Define direct and inverse Fourier transform of a function $F(\mathbf{r})$ in 2D position space $\mathbf{r} = \{x, y\}$ into function $\tilde{F}(\mathbf{k})$ in 2D wavevector space $\mathbf{k} = \{k_x, k_y\}$ as

$$\begin{aligned} \tilde{F}(\mathbf{k}) &= \int d^2\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}} F(\mathbf{r}) \\ F(\mathbf{r}) &= \int \frac{d^2\mathbf{k}}{(2\pi)^2}\, e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{F}(\mathbf{k}) \end{aligned}$$

where $d^2\mathbf{r} = dx\, dy$ and $d^2\mathbf{k} = dk_x\, dk_y$ and the integration is over the whole plane. Below, we also use polar coordinates $r = |\mathbf{r}|$, $\theta = \mathrm{atan2}(y, x)$ in position space and $k = |\mathbf{k}|$, $\chi = \mathrm{atan2}(k_y, k_x)$ in wavevector space.

Let $F(\mathbf{r}) = \delta(r-a)e^{-in\theta}$. The Fourier image is

$$\tilde{F}(\mathbf{k}) = \int d^2\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}} F(\mathbf{r}) = \int_0^\infty r\, dr \int_0^{2\pi} d\theta\, \delta(r-a) e^{-in\theta - ikr\cos(\theta-\chi)}$$

Using $\phi = \theta - \xi - \pi/2$, we get $\cos(\theta-\chi) = -\sin\phi$ and $\theta = \phi + \chi + \pi/2 \quad \Longrightarrow$

$$\tilde{F}(\mathbf{k}) = 2\pi a e^{-in\chi - in\pi/2} \int \frac{d\phi}{2\pi}\, e^{-in\phi + ika\sin\phi} = 2\pi a i^{-n} e^{-in\chi} J_n(ka)$$

We can use the same formula by substituting $\mathbf{r} \leftrightarrow \mathbf{k}$ and $i \leftrightarrow -i$ and taking care of coefficient $(2\pi)^2$. It summary,

$$\begin{aligned} F(\mathbf{r}) &\iff \tilde{F}(\mathbf{k}) \\ \frac{\delta(r-a)}{a} e^{in\theta} &\iff 2\pi i^{-n} J_n(ka) e^{in\chi} \qquad (14) \\ i^n J_n(pr) e^{in\theta} &\iff \frac{2\pi\delta(k-p)}{p} e^{in\chi} \qquad (15) \end{aligned}$$

3.2 Weber's First Integral

See *Abramowitz and Stegun* (1965, eq 11.4.28 with $\mu = 2, \nu = 0$).

The integral $I_1 = \int_0^\infty r\,dr\,J_0(pr)e^{-q^2r^2}$ is found by representing this integral in a 2D plane ($2\pi r\,dr = d^2\mathbf{r}$) and going to Fourier space $\mathbf{r} \to \mathbf{k}$.

$I_1 = \int d^2\mathbf{r}\,F_1(\mathbf{r})F_2(\mathbf{r})$, where $F_1(\mathbf{r}) = e^{-q^2r^2}$, $F_2(\mathbf{r}) = \frac{1}{2\pi}J_0(pr)$, where $r = |\mathbf{r}|$. We use the fact that $\int d^2\mathbf{r}\,F_1(\mathbf{r})F_2(\mathbf{r}) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\,\tilde{F}_1^*(\mathbf{k})\tilde{F}_2(\mathbf{k})$, where $\tilde{F}_{1,2}(\mathbf{k}) = \int d^2\mathbf{r}\,e^{-i\mathbf{k}\cdot\mathbf{r}}F_{1,2}(\mathbf{r})$ are the Fourier images. We have $\tilde{F}_1(\mathbf{k}) = \frac{\pi}{q^2}e^{-\frac{k^2}{4q^2}}$ and using the Fourier transform result from above, we obtain $\tilde{F}_2(\mathbf{k}) = \frac{\delta(k-p)}{p}$. Thus,

$$\int_0^\infty x\,dx\,J_0(px)e^{-q^2x^2} = \frac{1}{2q^2}e^{-\frac{p^2}{4q^2}} \tag{16}$$

3.3 Weber's Second Integral

$$I_2 = \int_0^\infty x\,dx\,J_n(ax)J_n(bx)e^{-q^2x^2}$$

From Graf's addition theorem (13) $\implies$ $J_n(x)J_n(y) = \int \frac{d\phi}{2\pi}\,J_0(z)e^{-in\phi}$ $\implies$

$J_n(ax)J_n(bx) = \int \frac{d\theta}{2\pi}\,J_0(x\sqrt{a^2+b^2-2ab\cos\theta})e^{-in\theta}$ $\implies$
$I_2 = \int \frac{d\theta}{2\pi}\,e^{-in\theta}\int_0^\infty x\,dx\,J_0(x\sqrt{a^2+b^2-2ab\cos\theta})e^{-q^2x^2}$ $\implies$ [use (16)] $\implies$

$I_2 = \int \frac{d\theta}{2\pi}\,\frac{1}{2q^2}\exp\left[-\frac{a^2+b^2-2ab\cos\theta}{4q^2} - in\theta\right] = \frac{1}{2q^2}e^{-\frac{a^2+b^2}{4q^2}}\int \frac{d\phi}{2\pi}\,\exp\left[i(-i\frac{ab}{2q^2})\sin\phi - in\phi + in\frac{\pi}{2}\right]$
(we substituted $\theta = \phi - \pi/2$)
$I_2 = \frac{1}{2q^2}e^{-\frac{a^2+b^2}{4q^2}}\,J_n(-i\frac{ab}{2q^2})i^n$
We use the definition of the modified Bessel function

$$I_n(x) = i^nJ_n(-ix) = i^{-n}J_n(ix)$$

$\implies$

$$\int_0^\infty x\,dx\,J_n(ax)J_n(bx)e^{-q^2x^2} = \frac{1}{2q^2}e^{-\frac{a^2+b^2}{4q^2}}I_n\left(\frac{ab}{2q^2}\right) \tag{17}$$

In particular, for $a = b$ we have

$$\int_0^\infty x\,dx\,J_n^2(ax)e^{-q^2x^2} = \frac{1}{2q^2}e^{-\frac{a^2}{2q^2}}I_n\left(\frac{a^2}{2q^2}\right) = \frac{1}{2q^2}\Lambda_n\left(\frac{a^2}{2q^2}\right)$$

where we introduced another function Λ_n whose properties are explored in more detail below in Section 5.

3.4 Lipschitz Integral

$I = \int_0^\infty e^{-ax} J_0(bx)\,dx = \int \frac{d\theta}{2\pi} \int_0^\infty e^{-ax+ibx\sin\theta}\,dx = \int \frac{d\theta}{2\pi}\frac{1}{a-ib\sin\theta}$
Substitute $z = e^{i\theta}$, integration is now around the unit circle in the complex z-plane:
$I = \frac{1}{2\pi}\oint \frac{dz/(iz)}{a-\frac{b}{2}(z-1/z)} = -\frac{1}{2\pi i}\frac{2}{b}\oint \frac{dz}{z^2-\frac{2a}{b}z-1} = -\frac{1}{2\pi i}\frac{2}{b}\oint \frac{dz}{(z-z_1)(z-z_2)}$, where $z_{1,2} = \frac{1}{b}(a \pm \sqrt{a^2+b^2})$.

Note that z_1 is outside the unit circle, z_2 is inside. We use the residue calculus and find $I = -\frac{2}{b}\frac{1}{(z_2-z_1)} = \frac{1}{\sqrt{a^2+b^2}}$.

$$\int_0^\infty e^{-ax} J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}} \tag{18}$$

4 Asymptotics

4.1 Taylor series expansion

We will derive the expansion of $J_n(x)$ in the vicinity of $x = 0$ using (4). Expand

$$e^{ix\sin\theta} = \sum_{m=0}^{\infty} \frac{x^m (i\sin\theta)^m}{m!} = \sum_{m=0}^{\infty} \frac{(e^{i\theta}-e^{-i\theta})^m}{m!}\left(\frac{x}{2}\right)^m$$

We use the binomial expansion

$$\frac{(e^{i\theta}-e^{-i\theta})^m}{m!} = \sum_{k=0}^{m} \frac{(-1)^k e^{i[(m-k)-k]\theta}}{(m-k)!k!}$$

The integral over θ "cuts out" only the exponents with the correct argument:

$$J_n(x) = \int \frac{d\theta}{2\pi} e^{ix\sin\theta - in\theta} = \sum_{m=0}^{\infty}\sum_{k=0}^{m} \delta_{n,m-2k} \frac{(-1)^k}{(m-k)!k!}\left(\frac{x}{2}\right)^m$$

Let us assume $n \geq 0$ (the formulas for $n < 0$ are obtained trivially using equation 2). After substitution $m = n + 2k$ instead of double summation we have summation in k only:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!}\left(\frac{x}{2}\right)^{n+2k} \tag{19}$$

4.2 Asymptotics for large x

Again, use (4). The phase $\phi = x\sin\theta - n\theta$ changes fast for large x, therefore we should use the stationary phase integration method. The stationary phase is at $\theta = \pm\theta_0$ found from $\phi'(\theta) = x\cos\theta - n = 0 \quad \Longrightarrow \quad \cos\theta_0 = n/x \ll 1$, we can use $\sin\theta_0 \approx 1$, $\theta_0 \approx \pi/2$ and $\phi(\theta_0) \approx x - n\pi/2$. The second derivative at θ_0 is $\phi''(\theta_0) = -x\sin\theta_0 \approx -x$. The contribution at $-\theta_0$ is a complex conjugate. Thus, we extend integration over $\zeta = \theta - \theta_0$ to infinite limits and get

$$
\begin{aligned}
J_n(x) &\approx \exp[i\phi(\theta_0)] \int_{-\infty}^{+\infty} \frac{d\zeta}{2\pi} \exp[i\phi''(\theta_0)\zeta^2/2] + \text{c.c.} \\
&= \exp[i(x - n\pi/2)] \int_{-\infty}^{+\infty} \frac{d\zeta}{2\pi} \exp[-ix\zeta^2/2] + \text{c.c.}
\end{aligned}
$$

Using gaussian integral $\int e^{-\zeta^2/(2\sigma^2)}\, d\zeta = \sqrt{2\pi}\sigma$ with $\sigma = \sqrt{1/(ix)}$, we find

$$
J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right), \quad x \to \infty \tag{20}
$$

5 Properties of $\Lambda_n(x)$

Introduce $\Lambda_n(x) = e^{-x} I_n(x)$. First, some properties of $I_n(x) = i^n J_n(-ix) = i^{-n} J_n(ix)$ which are derived elementarily from the corresponding properties of J_n:

$$
\begin{aligned}
x^2 I_n'' &= -x I_n' + (x^2 + n^2) I_n \\
I_n' &= \frac{1}{2}(I_{n-1} + I_{n+1}) \\
n I_n &= \frac{x}{2}(I_{n-1} - I_{n+1}) \\
I_{-n} &= I_n \\
\sum_n I_n(x) e^{in\theta} &= e^{x\cos\theta}
\end{aligned}
$$

From here, it follows that $\Lambda_n' = e^{-x}(I_n' - I_n)$. Some important properties of Λ_n:

$$
\sum_n \Lambda_n(x) = 1 \quad \text{(follows from the sum of } I_n \text{ above with } \theta = 0)
$$

$$\sum_n n\Lambda_n(x) \quad = \quad \frac{x}{2}\sum_n(\Lambda_{n-1}-\Lambda_{n+1}) = 0$$

5.1 Taylor series

Let us derive Taylor series expansion of Λ_n. We can assume $n \geq 0$ because otherwise $\Lambda_n = \Lambda_{-n}$. We start with (see equation 19)

$$e^{-x} = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}, \quad I_n(x) = \sum_{l=0}^{\infty}\frac{1}{l!\,(n+l)!}\left(\frac{x}{2}\right)^{2l+n}$$

Multiplying, we get

$$e^{-x}I_n(x) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-2)^k}{k!\,l!\,(n+l)!}\left(\frac{x}{2}\right)^{2l+n+k} = \sum_{p=n}^{\infty}A_p\left(\frac{x}{2}\right)^p$$

The goal is to find a tractable expression for

$$A_p = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\Bigg|_{\substack{k+l+m=p\\ m-l=n}} \frac{(-2)^k}{k!\,l!\,m!}$$

We use the polynomial expansion

$$(a+b+c)^p = \sum_{\substack{k,l,m=0\\ k+l+m=p}}^{\infty}\frac{p!}{k!\,l!\,m!}a^k b^l c^m$$

Take $a=-2$, need to single out term $b^l c^m$ such that $m-l=n$. Take $b=e^{i\theta}$, $c=e^{-i\theta}$, we get $b^l c^m = e^{-in\theta}$ which may be singled out by integration over θ of this expression multiplied by $e^{in\theta}$:

$$A_p = \frac{1}{p!}\int_0^{2\pi}(-2+e^{i\theta}+e^{-i\theta})^p e^{in\theta}\,\frac{d\theta}{2\pi}$$

Let us simplify using $-2+e^{i\theta}+e^{-i\theta} = -(2\sin\theta')^2$, with $\theta'=\theta/2$. Integration over θ' is from 0 to π but may be extended to $[0,2\pi]$ because of the symmetry of the expression under the integral. After dropping the dash from θ' and using $2\sin\theta = -i(e^{i\theta}-e^{-i\theta})$:

$$A_p = \frac{1}{p!}\int_0^{2\pi}(e^{i\theta}-e^{-i\theta})^{2p}e^{2in\theta}\,\frac{d\theta}{2\pi}$$

Use the binomial expansion:

$$(e^{i\theta} - e^{-i\theta})^{2p} = \sum_{j=0}^{2p} \frac{(2p)!\,(-1)^{2p-j}}{j!\,(2p-j)!} e^{i\theta(j-2p+j)}$$

From this expansion, the integration over θ only extracts the term $\propto e^{-2in\theta}$, i.e., $j = p - n$. Finally,

$$A_p = \frac{(2p)!\,(-1)^{p+n}}{p!\,(p-n)!\,(p+n)!}$$

and we have our final answer

$$\Lambda_n(x) = \sum_{k=n}^{\infty} \frac{(2k)!\,(-1)^{n+k}}{k!\,(k-n)!\,(k+n)!} \left(\frac{x}{2}\right)^k \tag{21}$$

5.2 Large x

We use (20) to find

$$\Lambda_n(x) = i^n J_n(-ix)e^{-x} \approx \sqrt{\frac{1}{2\pi x}}, \quad x \to \infty$$

6 Hankel functions

This is extra material which is used in completely different situations. In plasma physics, J_n is usually used to describe the field of a plane wave seen by an oscillating particle, using the expansion (3). On the other hand, Hankel functions $H_n^{(1,2)}(x)$ are usually used to describe waves in a cylindrical system (to which J_n may also have a limited applicability).

6.1 Cylindrical waves

We will heavily use notations from Subsection 3.1. Let us consider the Helmholtz (wave) equation in 2D. From equation (15) we see that the function $\Phi(\mathbf{r}) = i^n J_n(pr)e^{in\theta}$ in configuration $\mathbf{r}$-space becomes $\tilde{\Phi}(\mathbf{k}) = \frac{2\pi}{p}\delta(k - p)e^{in\chi}$ in the wave vector $\mathbf{k}$-space which satisfies the uniform Helmholtz equation $(-k^2 + p^2)\tilde{\Phi}(\mathbf{k}) = 0$ or $(\nabla^2 + p^2)\Phi(\mathbf{r}) = 0$. Notice that another way to show this is that by separating variables θ and r we get the differential equation (9). Let us consider a Helmholtz equation with sources:

$$(\nabla^2 + p^2)\Phi(\mathbf{r}) = -S(\mathbf{r})$$

or, in $\mathbf{k}$-domain,

$$(-k^2 + p^2)\tilde{\Phi}(\mathbf{k}) = -\tilde{S}(\mathbf{k})$$

(the minus sign is for convenience).

The simplest source would be a δ-function in the origin, however, it is cylindrically symmetric and can only generate the azimuthal harmonic with $n = 0$. As a more general source, let us take, for example, a thin (δ-function) source at distance a from origin, such as given by equation (14) (we switch $n \to -n$ for later convenience), with Fourier image

$$\tilde{S}(\mathbf{k}) = 2\pi i^{-n} J_n(ka) e^{-in\chi} = s(k) i^{-n} e^{-in\chi}$$

The solution in $\mathbf{r}$-domain is

$$\Phi(\mathbf{r}) = i^{-n} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{s(k) e^{i\mathbf{k}\cdot\mathbf{r} - in\chi}}{k^2 - p^2}$$

The azimuthal dependence of the answer will be $\Phi(\mathbf{r}) = \Phi(r)e^{-in\theta}$, so it is sufficient to find Φ at, say, $\mathbf{r} = r\hat{y}$ with $r > 0$, i.e. $\Phi(r) = \Phi(r\hat{y})e^{in(\pi/2)} = i^n\Phi(r\hat{y})$. Let us evaluate this integral in Cartesian coordinates k_x, k_y:

$$\Phi(r) = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{s(k) e^{i\mathbf{k}\cdot\mathbf{r} - in\chi}}{k^2 - p^2} = \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} F(k_x)$$

where $F(k_x)$ is the result of integration over k_y:

$$F(k_x) = \int_{-\infty}^{+\infty} \frac{dk_y}{2\pi} \frac{s(k) e^{ik_y r - in\chi}}{k_y^2 - (p^2 - k_x^2)}$$

where $k = \sqrt{k_x^2 + k_y^2}$ and $\chi = \mathrm{atan2}(k_y, k_x)$.

For the ease of integration in complex k_y plane, we assume $s(k)$ is an arbitrary function that is analytic in the upper half-plane ($\Im k_y > 0$) and also not very big so that $s(k)e^{ik_y r} \to 0$ at $\Im k_y \to +\infty$. For example, we may take the above expression for a thin source with $a \to 0$ (a small source) so using (19) $s(k) \approx 2\pi(n!2^n)^{-1}(ak)^n\,[1 + O(ak)] \propto k^n$. Note that for a finite-size source, $s(k) = 2\pi J_n(ka)$, at $k_y = iX$, $X \to +\infty$ asymptotically becomes, using $k \approx k_y = iX$:

$$s(k) \approx \sqrt{\frac{8\pi}{iXa}} \cos\left(iXa - \frac{n\pi}{2} - \frac{\pi}{4}\right) \approx i^n \sqrt{\frac{2\pi}{Xa}} e^{Xa}$$

so the product $s(k)e^{ik_y r} \sim e^{X(a-r)} \to 0$ only when $r > a$ (makes sense because the free waves are *outside* the source).

6.2 Outward waves: $H_n^{(1)}(x)$

To get an outward propagating wave solution, assume that p has a small positive imaginary part, $p \to p + i\Delta$ (i.e., the medium has a small absorption which makes the inward wave infinite at $r \to \infty$ and therefore the outward wave is the only one that we get). This gives us the rule for going around the pole at $k \approx p$. Since $r > 0$, we must close the integration contour in the upper plane, i.e., go counterclockwise around the pole at $k_y = \sqrt{p^2 - k_x^2}$ if $p > |k_x|$ and the pole at $k_y = i\sqrt{k_x^2 - p^2}$ when $p < |k_x|$. In both cases, we can write

$$F(k_x) = \frac{i}{2} \frac{s(p)e^{ipr\sin\chi - in\chi}}{p\sin\chi}$$

where

$$\begin{aligned} p\cos\chi &= k_x \\ p\sin\chi &= \begin{cases} \sqrt{p^2 - k_x^2} & \text{when } p > |k_x| \text{ or } |\cos\chi| < 1 \\ i\sqrt{k_x^2 - p^2} & \text{when } p < |k_x| \text{ or } |\cos\chi| > 1 \end{cases} \end{aligned}$$

i.e., $p\sin\chi$ is the k_y value at the pole. Changing the integration variable $dk_x = -p\sin\chi\, d\chi$ we finally get

$$\Phi(r) = -s(p)\frac{i}{2}\int_C \frac{d\chi}{2\pi}\, e^{ipr\sin\chi - in\chi}$$

where the contour C is such that $\cos\theta$ goes through the values $\{-\infty, -1, 0, 1, +\infty\}$ and $\sin\theta$ goes through the values $\{+i\infty, 0, 1, 0, +i\infty\}$. This corresponds to a contour in χ going through points $\{\pi - i\infty, \pi, \pi/2, 0, +i\infty\}$. For convenience, usually a contour C_1 is chosen which goes in the opposite direction. We define (compare to (4)!)

$$H_n^{(1)}(x) = 2\int_{C_1} \frac{d\theta}{2\pi}\, e^{ix\sin\theta - in\theta}, \quad C_1 = \{+i\infty, 0, \pi/2, \pi, \pi - i\infty\} \tag{22}$$

Thus

$$\Phi(\mathbf{r}) = s(p)\frac{i}{4}H_n^{(1)}(pr)e^{-in\theta}$$

In particular, for a point source $S(\mathbf{r}) = \delta(\mathbf{r})$ we have $n = 0$ and $s(p) = 1$ so the "outward" Green's function of Helmholtz equation is

$$G(\mathbf{r}) = \frac{i}{4}H_0^{(1)}(pr)$$

6.3 Inward waves: $H_n^{(2)}(x)$

If we look for inward waves, we set $p \to p - i\Delta$ and the pole in the upper half of k_y-plane is at $k_y = -\sqrt{p^2 - k_x^2}$ if $p > |k_x|$ and at $k_y = i\sqrt{k_x^2 - p^2}$ when $p < |k_x|$. We get the same expression for $F(k_x)$ but now

$$\begin{aligned} p\cos\chi &= k_x \\ p\sin\chi &= \begin{cases} -\sqrt{p^2 - k_x^2} & \text{when } p > |k_x| \text{ or } |\cos\chi| < 1 \\ i\sqrt{k_x^2 - p^2} & \text{when } p < |k_x| \text{ or } |\cos\chi| > 1 \end{cases} \end{aligned}$$

Using $dk_x = -p\sin\chi\, d\chi$ we get

$$\Phi(r) = -s(p)\frac{i}{2}\int_{C_2}\frac{d\chi}{2\pi}\, e^{ipr\sin\chi - in\chi}$$

where the contour C_2 is such that $\cos\theta$ goes through the values $\{-\infty, -1, 0, 1, +\infty\}$ and $\sin\theta$ goes through the values $\{+i\infty, 0, -1, 0, +i\infty\}$. This corresponds to a contour in χ going through points $\{\pi - i\infty, \pi, 3\pi/2, 2\pi, 2\pi + i\infty\}$. We define

$$H_n^{(2)}(x) = 2\int_{C_2}\frac{d\theta}{2\pi}\, e^{ix\sin\theta - in\theta}, \quad C_2 = \{\pi - i\infty, \pi, 3\pi/2, 2\pi, 2\pi + i\infty\} \quad (23)$$

and

$$\Phi(\mathbf{r}) = -s(p)\frac{i}{4}H_n^{(2)}(pr)e^{-in\theta}$$

The "inward" Green's function is

$$G(\mathbf{r}) = -\frac{i}{4}H_0^{(2)}(pr)$$

6.4 General properties

$H_n^{(1,2)}(x)$ satisfy the same recursion relations (5–6) and the differential equation (9) as $J_n(x)$ because they are defined through an integral similar to (4), although the contours of integration are different. Let us consider the symmetry relations analogous to equation (2) in more detail. In (22) for $-n$, change variable to $\alpha = \pi - \theta$ ($\theta = \pi - \alpha$). The contour C_1 changes to $-C_1$ (by minus we mean oppositely directed):

$$H_{-n}^{(1)}(x) = 2\int_{\alpha\in -C_1} \frac{d(-\alpha)}{2\pi}\, e^{ix\sin(\pi-\alpha)+in\pi-in\alpha} = e^{in\pi} H_n^{(1)}(x)$$

which is also valid for non-integer n if we postulate the definition (22). For integer n, $e^{in\pi} = (-1)^n$.

Changing the variable to $\alpha = -\theta^*$, and using $\sin\alpha^* = (\sin\alpha)^*$, we get (the contour C_1 for θ becomes $-C_2$ for α)

$$H_n^{(1)}(x) = 2\int_{\alpha\in -C_2} \frac{d(-\alpha^*)}{2\pi}\, e^{-ix\sin\alpha^*+in\alpha^*} = \left(H_n^{(2)}(x^*)\right)^*$$

In summary,

$$H_n^{(1,2)}(x) = (-1)^n H_{-n}^{(1,2)}(x) = \left(H_n^{(2,1)}(x^*)\right)^* \tag{24}$$

There is no relation of symmetry when changing $x \to -x$ because for $\Re x < 0$ the defining integral (22,23) becomes infinite.

We notice that $C_1 + C_2 = [0, 2\pi]$ (the parts going over complex θ cancel, also due to 2π periodicity). Thus,

$$J_n(x) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{ix\sin\theta - in\theta} = \frac{1}{2}\left(H_n^{(1)}(x) + H_n^{(2)}(x)\right)$$

so $J_n(x) = \Re H_n^{(1,2)}(x)$ for real x. We may introduce Bessel functions of second kind (Weber or Neumann functions) $Y_n(x)$ which are real for real x:

$$\begin{aligned} H_n^{(1)}(x) &= J_n(x) + iY_n(x) \\ H_n^{(2)}(x) &= J_n(x) - iY_n(x) \end{aligned}$$

But they are only useful as linearly independent from $J_n(x)$ solutions of Helmholtz equation (or Bessel equation) and don't have extra physical meaning compared to $H_n^{(1,2)}(x)$. Moreover, $Y_n(x)$ are infinite at $x \to 0$. This means there is no Taylor expansion at $x \to 0$.

6.5 Asymptotics

We noted that there is no Taylor expansion of $H_n^{(1,2)}$ at $x \to 0$. However, we may try to find the asymptotic expansion or at least the biggest term. Because of the symmetries, it is sufficient to consider only $H_n^{(1)}$ and $n \geq 0$. For $n = 0$ we may use the Green's function at $pr \to 0$, i.e., take $p = 0$ in the Helmholtz equation and compare to the Poisson equation Green's function

$$G_{\text{Poisson}}(\mathbf{r}) = -\frac{\log r}{2\pi} + const = -\frac{\log(pr/2)}{2\pi} + const_2 \approx \frac{i}{4} H_0^{(1)}(pr)$$

from where $H_0^{(1)}(x) \approx i(2/\pi)\log(x/2) + const$. We may neglect the constant at very small x.

For $n \geq 1$, write out (22) explicitly, while changing the integration variable so that $\theta = it$ on the interval $\theta \in [i\infty, 0]$ and $\theta = \pi - it$ on the interval $\theta \in [\pi, \pi - i\infty]$:

$$H_n^{(1)}(x) = -\frac{i}{\pi}\int_0^\infty e^{-x\sinh t + nt}\,dt + \frac{1}{\pi}\int_0^\pi e^{ix\sin\theta - in\theta}\,d\theta - \frac{i(-1)^n}{\pi}\int_0^\infty e^{-x\sinh t - nt}\,dt$$

The middle term is finite for $x = 0$ and thus does not contribute to the infinite value. Regarding the last term we notice that for $x, t > 0$ we have $\sinh t > t$, $e^{-x\sinh t - nt} < e^{-(x+n)t}$ and therefore

$$\int_0^\infty e^{-x\sinh t - nt}\,dt < \frac{1}{x+n} < \frac{1}{n} \text{ (finite)}$$

In the first term, change the variable to $u = x\sinh t$ or $t = \log\left[(u + \sqrt{u^2 + x^2})/x\right]$:

$$\int_0^\infty e^{-x\sinh t + nt}\,dt = \frac{1}{x^n}\int_0^\infty \frac{e^{-u}(u + \sqrt{u^2 + x^2})^n}{\sqrt{u^2 + x^2}}\,du$$

We may set $x = 0$ in the integral because it is finite:

$$\int_0^\infty e^{-x\sinh t + nt}\,dt \approx \frac{1}{x^n}\int_0^\infty \frac{e^{-u}(2u)^n}{u}\,du = (n-1)!\left(\frac{x}{2}\right)^{-n}$$

Collecting everything, we have for the biggest term

$$H_n^{(1)}(x) \approx \begin{cases} i\left(\frac{2}{\pi}\right)\log\left(\frac{x}{2}\right), & n = 0 \\ -i\frac{(n-1)!}{\pi}\left(\frac{x}{2}\right)^{-n}, & n > 0 \end{cases} \tag{25}$$

As expected, it is purely imaginary because the real part ($= J_n(x)$) is finite.

Asymptotics of $H_n^{(1,2)}(x)$ at $x \to \infty$ is obtained in the same way as we did for $J_n(x)$ above in equation (20). This time, we have only one stationary phase point at $\theta_0 = \arccos(n/x) \approx \pi/2$ on contour C_1 (or $\theta_0 = 2\pi - \arccos(n/x) \approx 3\pi/2$ on contour C_2). The value at large x is therefore

$$H_n^{(1,2)}(x) \approx \sqrt{\frac{2}{\pi x}} \exp\left[\pm i\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right], \quad x \to \infty \tag{26}$$

which makes it obvious that they represent outward and inward propagating waves. The non-oscillating contributions from intervals $\theta \in [+i\infty, 0]$ and $\theta \in [\pi, \pi - i\infty]$ contribute a smaller amount since $e^{ix\sin\theta - in\theta} \approx e^{-x|\theta|}$ with integral (from both intervals) $\approx 2/(\pi x)$.

References

Abramowitz, M., and I. A. Stegun (1965), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover.