Error functions

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1 Error function $\text{erf } x$ and complementary error function $\text{erfc } x$

*(Gauss) error function* is

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt \quad (1)$$

and has properties

$$\begin{align*}
\text{erf } (-\infty) &= -1, & \text{erf } (+\infty) &= 1 \\
\text{erf } (-x) &= -\text{erf } (x), & \text{erf } (x^*) &= [\text{erf } (x)]^*
\end{align*}$$

where the asterisk denotes complex conjugation. **Complementary error function** is defined as

$$\text{erfc } x = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt = 1 - \text{erf } x \quad (2)$$

Note also that

$$\frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2} dt = 1 + \text{erf } x$$

Another useful formula:

$$\int_{0}^{x} e^{-\frac{t^2}{2\sigma^2}} dt = \sqrt{\frac{\pi}{2\sigma}} \text{erf } \left[ \frac{x}{\sqrt{2\sigma}} \right]$$

Some Russian authors (e.g., Mikhailovskiy, 1975; Bogdanov et al., 1976) call $\text{erf } x$ a *Cramp function.*
2 Faddeeva function $w(x)$

Faddeeva (or Fadeeva) function $w(x)$ (Fadeeva and Terent’ev, 1954; Poppe and Wijers, 1990) does not have a name in Abramowitz and Stegun (1965, ch. 7). It is also called complex error function (or probability integral) (Weideman, 1994; Baumjohann and Treumann, 1997, p. 310) or plasma dispersion function (Weideman, 1994). To avoid confusion, we will reserve the last name for $Z(x)$, see below. Some Russian authors (e.g., Mikhailovskiy, 1975; Bogdanov et al., 1976) call it a (complex) Cramp function and denote as $W(x)$. Faddeeva function is defined as

$$w(x) = e^{-x^2} \left( 1 + \frac{2i}{\sqrt{\pi}} \int_0^x e^{t^2} dt \right) = e^{-x^2} \left[ 1 + \text{erf}(ix) \right] = e^{-x^2} \text{erfc}(-ix) \quad (3)$$

Integral representations:

$$w(x) = \frac{i}{\pi} \int_{-\infty}^{\infty} e^{-t^2} dt \frac{dx}{x-t} = \frac{2ix}{\pi} \int_0^{\infty} e^{-t^2} dt \frac{dx}{x^2-t^2} \quad (4)$$

where $\Im x > 0$. These integral representations can be converted to (3) using

$$\frac{1}{x+i\Delta-t} = -2i \int_0^{\infty} e^{2i(x+i\Delta-t)u} du \quad (5)$$

3 Plasma dispersion function $Z(x)$

Plasma dispersion function $Z(x)$ (Fried and Conte, 1961) is also called Fried-Conte function (Baumjohann and Treumann, 1997, p. 268). In the book by Mikhailovskiy (1975), notation is $Z_{\text{Mikh}}(x) \equiv xZ(x)$, which may be a source of confusion. Plasma dispersion function is defined as:

$$Z(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-t^2}}{t-x} dt \quad (6)$$

for $\Im x > 0$, and its analytic continuation to the rest of the complex $x$ plane (i.e., to $\Im x \leq 0$).

Note that $Z(x)$ is just a scaled $w(x)$, i.e.

$$Z(x) \equiv i\sqrt{\pi} w(x)$$
We see that
\[ Z(x) = 2ie^{-x^2} \int_{-\infty}^{ix} e^{-t^2} dt = i\sqrt{\pi}e^{-x^2}[1 + \text{erf}(ix)] = i\sqrt{\pi}e^{-x^2}\text{erfc}(-ix) \] (7)

One can define \( \bar{Z} \) which is given by the same equation (6), but for \( \Im x < 0 \), and its analytic continuation to \( \Im x \geq 0 \). It is related to \( Z(x) \) as
\[ \bar{Z}(x) = Z^*(x^*) = Z(x) - 2i\sqrt{\pi}e^{-x^2} = -Z(-x) \]

4  (Jackson) function \( G(x) \)

Another function useful in plasma physics was introduced by Jackson (1960) and does not (yet) have a name (to my knowledge):
\[ G(x) = 1 + i\sqrt{\pi}xw(x) = 1 + xZ(x) = -Z'(x)/2 \] (8)

Integral representation:
\[ G(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{te^{-t^2}}{t - x} dt \] (9)

where again \( \Im x > 0 \).

5  Other definitions

5.1  Dawson integral

Dawson integral \( F(x) \) (Abramowitz and Stegun, 1965, eq. 7.1.17), also denoted as \( S(x) \) by Stix (1962, p. 178) and as daw \( x \) by Weideman (1994), is defined as:
\[ F(z) = e^{-z^2} \int_{0}^{z} e^{t^2} dt = (-Z(x) + i\sqrt{\pi}e^{-x^2})/2 = xY(x) \]

(see also the definition of \( Y(x) \) below).

5.2  Fresnel functions

Fresnel functions (Abramowitz and Stegun, 1965, ch. 7) \( C(x), S(x) \) are defined by
\[ C(x) + iS(x) = \int_{0}^{x} e^{\pi i t^2/2} dt \]
5.3 (Sitenko) function $\varphi(x)$

Another function (Sitenko, 1982, p. 24) is $\varphi(x)$, defined only for real arguments:

$$\varphi(x) = 2x e^{-x^2} \int_0^x e^{t^2} \, dt$$

so that

$$G(x) = 1 - \varphi(x) + i\sqrt{\pi} x e^{-x^2}$$

5.4 Function $Y(x)$ of Fried and Conte (1961)

Fried and Conte (1961) introduce

$$Y(x) = \frac{e^{-x^2}}{x} \int_0^x e^{t^2} \, dt$$

so that for real argument

$$Z(x) = i\sqrt{\pi} e^{-x^2} - 2x Y(x)$$

6 Asymptotic formulas

6.1 For $|x| \ll 1$ (series expansion)

See Abramowitz and Stegun (1965, 7.1.8):

$$w(x) = \sum_{n=0}^{\infty} \frac{(ix)^n}{\Gamma \left( \frac{n}{2} + 1 \right)}$$

The even terms give $e^{-x^2}$. To collect odd terms, note that for $n \geq 0$:

$$\Gamma \left( n + \frac{3}{2} \right) = \frac{\sqrt{\pi} (2n + 1)!!}{2^n}$$

$$\Gamma \left( n + \frac{1}{2} \right) = \frac{\sqrt{\pi} (2n - 1)!!}{2^n}$$

where we define $(2n + 1)!! = 1 \cdot 3 \cdot 5 \cdots (2n + 1)$ and $(-1)!! = 1$. We have

$$Z(x) = i\sqrt{\pi} e^{-x^2} - 2x \sum_{n=0}^{\infty} \frac{(-2x^2)^n}{(2n + 1)!!}$$
The first few terms are

\[ w(x) \approx e^{-x^2} + \frac{2ix}{\sqrt{\pi}} \left( 1 - \frac{2x^2}{3} + \frac{4x^4}{15} - \ldots \right) \]  \hspace{1cm} (19)

\[ Z(x) \approx i\sqrt{\pi}e^{-x^2} - 2x \left( 1 - \frac{2x^2}{3} + \frac{4x^4}{15} - \ldots \right) \]  \hspace{1cm} (20)

The Jackson function (8) also has a nice expansion

\[ G(x) = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(ix)^n}{\Gamma\left(\frac{n+1}{2}\right)} \]  \hspace{1cm} (21)

\[ = i\sqrt{\pi}xe^{-x^2} + \sum_{n=0}^{\infty} \frac{(-2x^2)^n}{(2n-1)!!} \]  \hspace{1cm} (22)

\[ \approx 1 + i\sqrt{\pi}x - 2x^2 + \ldots \]  \hspace{1cm} (23)

6.2 For \(|x| \gg 1\)

These formulas are valid for \(-\pi/4 < \arg x < 5\pi/4\) (Abramowitz and Stegun, 1965, 7.1.23), i.e., around positive imaginary axis:

\[ w(x) = \frac{i}{\sqrt{\pi}x} \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2x^2)^m} \]  \hspace{1cm} (24)

\[ Z(x) = -\frac{1}{x} \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2x^2)^m} \]  \hspace{1cm} (25)

\[ G(x) = -\sum_{m=1}^{\infty} \frac{(2m-1)!!}{(2x^2)^m} \]  \hspace{1cm} (26)

7 Gordeyev’s integral

Gordeyev’s integral \(G_{\nu}(\omega, \lambda)\) (Gordeyev, 1952) is defined as

\[ G_{\nu}(\omega, \lambda) = \omega \int_{0}^{\infty} \exp \left[ i\omega t - \lambda(1 - \cos t) - \frac{\nu t^2}{2} \right] dt, \quad \Re \nu > 0 \]  \hspace{1cm} (27)
and is calculated in terms of the plasma dispersion function $Z$ (Paris, 1998):

$$G_\nu(\omega, \lambda) = -\frac{i\omega}{\sqrt{2}\nu} e^{-\lambda} \sum_{n=-\infty}^{\infty} I_n(\lambda)Z\left(\frac{\omega - n}{\sqrt{2}\nu}\right)$$

(28)

8 Plasma permittivity

The dielectric permittivity of hot (Maxwellian) plasma is (Jackson, 1960)

$$\epsilon(\omega, \mathbf{k}) = 1 + \sum_s \Delta \epsilon_s = 1 + \sum_s \frac{1}{k^2 \lambda_s^2} G(x_s)$$

(29)

The summation is over charged species. For each species, we have introduced the Debye length

$$\lambda = \sqrt{\frac{\epsilon_0 T}{Nq^2}} = \frac{v}{\Pi}$$

where $v = \sqrt{T/m}$ is the thermal velocity and $\Pi = \sqrt{Nq^2/(m\epsilon_0)}$ is the plasma frequency; and

$$x = \frac{\omega - (\mathbf{k} \cdot \mathbf{u})}{\sqrt{2}kv}$$

where $\mathbf{u}$ is the species drift velocity.

For warm components, $x \gg 1$, we have

$$\Delta \epsilon = -\frac{\Pi^2}{[\omega - (\mathbf{k} \cdot \mathbf{u})]^2}\left(1 + \frac{3k^2v^2}{[\omega - (\mathbf{k} \cdot \mathbf{u})]^2}\right)$$

The dispersion relation for plasma oscillations is obtained by equating $\epsilon = 0$. For example, for warm electron plasma at rest,

$$\epsilon = 1 - \frac{\Pi^2}{\omega^2}\left(1 + \frac{3k^2v^2}{\omega^2}\right)$$

and we have the dispersion relation (for $\omega \approx \Pi$):

$$\omega^2 = \Pi^2 + 3k^2v^2 = \Pi^2 + k^2\langle v^2 \rangle$$
9 Ion acoustic waves

Assume that for electrons, $\omega \ll kv$ but for ions still $\omega \gg kV$ (we’ll see later that in means $T_i \ll T_e$). For $x \ll 1$, we use $G(x) \approx 1 + i\sqrt{\pi x}$:

$$\epsilon = 1 + \frac{\Pi_e^2}{k^2v^2} \left(1 + i\sqrt{\pi} \frac{\omega}{\sqrt{2kv}}\right) - \frac{\Pi_i^2}{\omega^2} \left(1 + \frac{3k^2V^2}{\omega^2}\right)$$  (30)

where $v$ and $V$ are thermal velocities of electrons and ions, respectively. From $\epsilon = 0$ we have

$$1 + \frac{3k^2V^2}{\omega^2} = \frac{\omega^2}{k^2v_s^2} \left(1 + k^2\lambda_e^2 + i\sqrt{\frac{\pi}{2\Pi_e k\lambda_e}}\right)$$

where $v_s = \lambda_e\Pi_i = \sqrt{ZT_e/M}$. If we neglect $V$ and imaginary part, then we get

$$\omega = \frac{kv_s}{\sqrt{1 + k^2\lambda_e^2}}$$  (31)

For long wavelengths, it reduces to the usual relation $\omega = kv_s$. If we substitute this into the imaginary part, we get

$$\omega \approx \frac{kv_s}{\sqrt{1 + i\sqrt{\frac{\pi Zm}{2M}}}}$$  (32)

The attenuation coefficient for the ion-acoustic waves is small:

$$\gamma = -\Im \omega = kv_s\sqrt{\frac{\pi Zm}{8M}} \ll \Re \omega$$

Neglecting $V$ is equivalent to $kV \ll \omega$, i.e., $V \ll v_s$ or $T_i \ll T_e$. Otherwise, the ion-acoustic waves do not exist.

10 Fourier transform of $Z(x)$

In the plasma dielectric permittivity expression, the argument of $G(x) = -Z'(x)$ is $x = [\omega - (k \cdot u)]/\sqrt{2kv}$. Thus, the plasma dispersion function is usually applied in the frequency domain. The argument is dimensionless, but is proportional to $\omega$. Let us consider an inverse Fourier transform of $\tilde{F}(\omega) =$
\[ Z(\omega/\sqrt{2\Omega}), \text{ where } \Omega \text{ is a parameter which has the same dimensionality as } \omega. \] (In the plasma dielectric permittivity expression, we have \( \Omega \equiv kv \).)

Quick reminder of the time-frequency and space-wavevector Fourier transforms:

\[
\tilde{X}(\omega, k) = \iiint X(t, r)e^{i\omega t - ik r} dt d^3 r
\]

\[
X(t, r) = \iiint \tilde{X}(\omega, k)e^{-i\omega t + ik r} \frac{d\omega}{2\pi} \frac{d^3 k}{(2\pi)^3}
\]

where integrals are from \(-\infty\) to \(+\infty\).

Using expression (6), we have

\[
\tilde{F}(\omega) = Z\left(\frac{\omega}{\sqrt{2\Omega}}\right) = \frac{2\pi}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\frac{\omega'^2}{2\Omega}} \frac{d\omega'}{2\pi}
\]

where the condition that \( \Im \omega > 0 \) is implemented by adding a small imaginary part to \( \omega \), i.e., \( \omega \to \omega + i\Delta \). We notice that the above expression is a convolution which simply gives a product in the \( t \)-domain:

\[
\tilde{F}(\omega) = \int_{-\infty}^{+\infty} \tilde{F}_1(\omega') \tilde{F}_2(\omega - \omega') \frac{d\omega'}{2\pi} \implies F(t) = F_1(t) F_2(t)
\]

where

\[
\tilde{F}_1(\omega) = 2\pi \frac{1}{\sqrt{2\pi\Omega}} e^{-\frac{\omega^2}{2\Omega}} \implies F_1(t) = e^{-\frac{\Omega^2 t^2}{2}}
\]

and

\[
\tilde{F}_2(\omega) = \sqrt{2i\Omega} \frac{i}{\omega + i\Delta} \implies F_2(t) = \sqrt{2i\Omega} H(t)
\]

where \( H(t) \) is the Heaviside (step) function, defined to be \( H(t) = 0 \) for \( t < 0 \) and \( H(t) = 1 \) for \( t > 0 \). (The value at \( t = 0 \) is not important, but most often is assumed to be \( 1/2 \).) The last inverse Fourier transform is accomplished by using the usual technique of integrating over a closed contour in the plane of complex \( \omega \) around the pole at \(-i\Delta\) and taking a residue. Note that the Fourier transform between \( F_2(t) \) and \( \tilde{F}_2(\omega) \) illuminates the physical sense of the trick used in equation (5).

Thus,

\[
\tilde{F}(\omega) = Z\left(\frac{\omega}{\sqrt{2\Omega}}\right) \implies F(t) = \sqrt{2i\Omega} H(t) e^{-\frac{\Omega^2 t^2}{2}}
\]
The derivative $G(x) = -Z'(x)/2$ is found by using $d/d\omega \rightarrow it$:

$$
\tilde{F}_G(\omega) = G \left( \frac{\omega}{\sqrt{2}\Omega} \right) = -\frac{\Omega}{\sqrt{2}} \frac{d}{d\omega} \tilde{F}(\omega) \quad \Rightarrow \quad F_G(t) = \Omega^2 tH(t)e^{-\frac{\Omega^2 t^2}{2}}
$$

The shift by $\Delta \omega = (k \cdot u)$ is accomplished using the property

$$
\tilde{F}(\omega - \Delta \omega) \quad \Rightarrow \quad e^{-i\Delta \omega t} F(t)
$$

Finally, we have the dielectric permittivity in time-wavevector domain:

$$
\epsilon(t, k) = \delta(t) + \sum_s \Delta \epsilon_s(t, k) = \delta(t) + tH(t) \sum_s \Pi^2 e^{-i\frac{k_s^2 v^2 t^2}{2} - i(k \cdot u_s) t}
$$

where the delta function is obtained from transforming “1”. The same answer may be obtained from the first principles by calculating the polarization created by a delta-function electric field in the time-space domain and converting $r \rightarrow k$. But this is a completely different topic.

References

Abramowitz, M., and I. A. Stegun (1965), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover.


