Fokker-Planck Equation (FPE)

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February 17, 2005

1 Introduction

Consider a distribution of particles $f(x_1, \ldots, x_n, t)$ in n-dimensional space with coordinates x_k which also is a function of time t. Its evolution is described by the Fokker-Planck equation (FPE), the derivation of which is beautifully explained in [1]. Generalizing this derivation to the case when distribution function depends on several variables, we get

$$\partial_t f = \sum_{l=1}^{\infty} (-1)^l \sum_{k_1,\ldots,k_l} \partial_{x_{k_1}} \cdots \partial_{x_{k_l}} (f D^{(l)}_{k_1,\ldots,k_l}) \tag{1}$$

where ∂ denotes the partial derivative in respect to the variable in the subscript. The tensors $D^{(l)}_{k_1,\ldots,k_l}$ are

$$D^{(l)}_{k_1,\ldots,k_l} = \frac{1}{l!} \lim_{\Delta t \to \text{`}0\text{'}} \frac{\langle \Delta x_{k_1} \cdots \Delta x_{k_l} \rangle}{\Delta t}$$

where Δx_i are the changes in variables x_i in time Δt due to random collisions and other processes. The limit $\Delta t \to$ ‘0’ means that we let Δt become small compared to any macroscopic time scale, while it still remains large compared with time intervals between collisions. The angular brackets $\langle \cdot \rangle$ denote the statistical (or ensemble) average. The FPE is understood as (1) truncated after $l = 2$.

2 FPE in momentum space and interpretation of coefficients D

We consider FPE in momentum space,

$$\frac{\partial f(\mathbf{p})}{\partial t} = -\nabla_{\mathbf{p}}[\mathbf{D}^{(1)} f(\mathbf{p})] + \nabla_{\mathbf{p}}[\nabla_{\mathbf{p}}(\overleftrightarrow{\mathsf{D}}^{(2)} f(\mathbf{p}))] \tag{2}$$

where

$$\mathbf{D}^{(1)} = \lim_{\Delta t \to `0'} \frac{\langle \Delta \mathbf{p} \rangle}{\Delta t}$$

and $\overleftrightarrow{\mathsf{D}}^{(2)}$ is a second rank tensor

$$\overleftrightarrow{\mathsf{D}}^{(2)} = \frac{1}{2} \lim_{\Delta t \to `0'} \frac{\langle \Delta \mathbf{p} \Delta \mathbf{p}^T \rangle}{\Delta t}$$

Here $\Delta \mathbf{p} \Delta \mathbf{p}^T$ is a dyadic product of vector $\Delta \mathbf{p}$ with itself. In the second term of (2), the inner $\nabla_{\mathbf{p}}$ is the divergence of a tensor (the result of which is a vector).

In general, $\mathbf{D}^{(1)}$ is due to changes in particle momenta due to both external force and random collisions. Without external force, it is usually directed along $(-\mathbf{p})$, and slows the particle down. However, the decrease of average vector momentum can be due to spreading in particle directions (which is described by the term with $\overleftrightarrow{\mathsf{D}}^{(2)}$) and thus not equal to the decrease of the absolute value of momentum vector. Since $\Delta(p^2) = 2\mathbf{p} \cdot \Delta \mathbf{p} + (\Delta p_x)^2 + (\Delta p_x)^2 + (\Delta p_z)^2$, we have an effective "force" (the dynamic friction coefficient)

$$\mathbf{F}_1 = \hat{p} \frac{\langle \Delta |\mathbf{p}| \rangle}{\Delta t} = \frac{\hat{p}}{2|\mathbf{p}|} \frac{\langle \Delta (p^2) \rangle}{\Delta t} = \mathbf{D}^{(1)} + \frac{\hat{p}}{|\mathbf{p}|} \mathrm{Tr}\, \overleftrightarrow{\mathsf{D}}^{(2)}$$

where $\hat{p}$ is the unit vector along $\mathbf{p}$ and Tr denotes the trace operation.

Equation (2) is usually rewritten as

$$\frac{\partial f(\mathbf{p})}{\partial t} = -\nabla_{\mathbf{p}}[(\mathbf{D}^{(1)} - \nabla_{\mathbf{p}} \overleftrightarrow{\mathsf{D}}^{(2)}) f(\mathbf{p})] + \nabla_{\mathbf{p}}[\overleftrightarrow{\mathsf{D}}^{(2)} \nabla_{\mathbf{p}} f(\mathbf{p})] \tag{3}$$

Thus, $\overleftrightarrow{\mathsf{D}}^{(2)} \nabla_{\mathbf{p}} f(\mathbf{p})$ can be interpreted as a flux of particles due to diffusion (in momentum space) with a diffusion coefficient $\overleftrightarrow{\mathsf{D}}^{(2)}$, and

$$\mathbf{F}_2 = \mathbf{D}^{(1)} - \nabla_{\mathbf{p}} \overleftrightarrow{\mathsf{D}}^{(2)}$$

as the force acting on particles. If the diffusion does not change the absolute value $p = |\mathbf{p}|$, then both interpretations of the force are the same, $\mathbf{F}_1 = \mathbf{F}_2$, or

$$-\nabla_{\mathbf{p}}\overleftrightarrow{\mathsf{D}}^{(2)} = \frac{\hat{p}}{|\mathbf{p}|}\mathrm{Tr}\,\overleftrightarrow{\mathsf{D}}^{(2)} \tag{4}$$

3 Calculation of D due to small-angle collisions with elastic fixed centers

Let the momentum before collision be $\mathbf{p} = p\hat{n}$ and after collision $\mathbf{p}' = p\hat{n}'$. The absolute value stays the same in elastic collisions. The change of momentum is

$$\Delta\mathbf{p} = p(\hat{n}' - \hat{n})$$

Then the needed coefficients are

$$\begin{aligned}\mathbf{D}^{(1)} &= p\frac{\langle(\hat{n}' - \hat{n})\rangle}{\Delta t}\\ \overleftrightarrow{\mathsf{D}}^{(2)} &= \frac{p^2}{2}\frac{\langle(\hat{n}' - \hat{n})(\hat{n}' - \hat{n})^T\rangle}{\Delta t}\end{aligned}$$

If we choose coordinate system so that $\mathbf{p} = p\hat{z}$, then

$$\hat{n}' - \hat{n} = \hat{z}(1 - \cos\theta) - (\hat{x}\cos\phi + \hat{y}\sin\phi)\sin\theta$$

Here θ, ϕ give the direction of the scattering Ω. To find the average over Ω, we take into account axial symmetry to average over ϕ, so that $\langle\sin^2\phi\rangle = \langle\cos^2\phi\rangle = 1/2$ and $\langle\sin\phi\cos\phi\rangle = 0$. We get

$$\langle\hat{n}' - \hat{n}\rangle_\Omega = -\langle 1 - \cos\theta\rangle\hat{n} \tag{5}$$

$$\langle(\hat{n}' - \hat{n})(\hat{n}' - \hat{n})^T\rangle_\Omega = \frac{1}{2}\langle\sin^2\theta\rangle\overleftrightarrow{\mathsf{I}}_\perp + \langle(1 - \cos\theta)^2\rangle\hat{n}\hat{n}^T \tag{6}$$

where we introduced the perpendicular projection operator

$$\overleftrightarrow{\mathsf{I}}_\perp = \overleftrightarrow{\mathsf{I}} - \hat{n}\hat{n}^T$$

($\overleftrightarrow{\mathsf{I}}$ is the unit operator). It has the following properties:

$$\overleftrightarrow{\mathsf{I}}_\perp\hat{n} = 0 \tag{7}$$

$$\mathrm{Tr}\,\overleftrightarrow{\mathsf{I}}_\perp = 2 \tag{8}$$

$$\overleftrightarrow{\mathsf{I}}_\perp = p\nabla_{\mathbf{p}}\hat{n}^T \tag{9}$$

where in the last equation a diadic product is taken. (**Proof:** $\partial_k p_l = \delta_{kl} = \partial_k(p\hat{n}_l) = p\partial_k\hat{n}_l + \hat{n}_k\hat{n}_l$, where we used $\partial_k p = \hat{n}_k$.) The equation (9) can be used to find the divergence $\nabla_{\mathbf{p}}\overleftrightarrow{\mathsf{I}}_\perp = -\nabla_{\mathbf{p}}(\hat{n}\hat{n}^T)$, $\sum_k \partial_k(\hat{n}_k\hat{n}_l) = (\overleftrightarrow{\mathsf{I}}_\perp\hat{n})_l/p + \hat{n}_l\mathrm{Tr}\,\overleftrightarrow{\mathsf{I}}_\perp/p = 2\hat{n}_l/p$:

$$\nabla_{\mathbf{p}}\overleftrightarrow{\mathsf{I}}_\perp = -\frac{2\hat{n}}{p}$$

and

$$\nabla_{\mathbf{p}}(f(p)\overleftrightarrow{\mathsf{I}}_\perp) = -f(p)\frac{2\hat{n}}{p}$$

for arbitrary function $f(p)$. Under the assumption of small-angle collisions equations (5–6) become

$$\begin{aligned} \langle \hat{n}' - \hat{n} \rangle_\Omega &\approx -\frac{\langle\theta^2\rangle}{2}\hat{n} && (10) \\ \langle (\hat{n}' - \hat{n})(\hat{n}' - \hat{n})^T \rangle_\Omega &\approx \frac{\langle\theta^2\rangle}{2}\overleftrightarrow{\mathsf{I}}_\perp && (11) \end{aligned}$$

where we neglected 4-th power of θ and higher.

The small-angle collisions are best described in terms of *momentum transfer frequency* ν_m, since the total cross-section, and therefore the collision frequency $\nu = 1/\Delta t$ can be infinite (e.g., for Coulomb potential). It is defined as

$$\nu_m = \frac{\langle 1 - \cos\theta \rangle}{\Delta t} \approx \frac{1}{2}\frac{\langle\theta^2\rangle}{\Delta t}$$

From (10–11) we get

$$\begin{aligned} \mathbf{D}^{(1)} &= -\nu_m\mathbf{p} \\ \overleftrightarrow{\mathsf{D}}^{(2)} &= D(p)p^2\overleftrightarrow{\mathsf{I}}_\perp \end{aligned}$$

where $D(p) = \nu_m/2$ can be considered as the angular diffusion coefficient. Note that any $\overleftrightarrow{\mathsf{D}}^{(2)} \sim \overleftrightarrow{\mathsf{I}}_\perp$, such as this one, conserves the absolute value of the momentum. Therefore, (4) is valid:

$$\nabla_{\mathbf{p}}\overleftrightarrow{\mathsf{D}}^{(2)} = -2\mathbf{p}D(p) = -\frac{\mathbf{p}}{p^2}\mathrm{Tr}\,\overleftrightarrow{\mathsf{D}}^{(2)} = -\nu_m\mathbf{p}$$

and the dynamic friction is $\mathbf{F} = \mathbf{D}^{(1)} - \nabla_{\mathbf{p}}\overleftrightarrow{\mathsf{D}}^{(2)} = 0$, as expected for elastic collisions.

4 Nonrelativistic elastic collisions with centers of finite mass

Consider collisions with centers of mass M, assuming that f describes particles of mass m. Example of this is electron distribution in an ambient gas of molecules. We also will assume that molecules move with speed $\mathbf{V}$, which has properties $\langle \mathbf{V} \rangle = 0$, and $\langle V^2 \rangle = 3T/M$, where T is an effective molecule temperature. We assume nonrelativistic motion here, so that to get momenta in the a differente reference frame (RF), e.g., the center-of-mass reference frame (CM), we use vector addition of velocities. We will write down the FP equation for the velocities $\mathbf{v}$ and $v = |\mathbf{v}|$.

The center of mass velocity is

$$\mathbf{U}_{\mathrm{CM}} = \frac{m\mathbf{v} + M\mathbf{V}}{M+m}$$

In the CM RF, the initial velocity is

$$\mathbf{v}_{\mathrm{CM}} = \mathbf{v} - \mathbf{U}_{\mathrm{CM}} = v_{\mathrm{CM}}\hat{n}$$

where now $\hat{n}$ is the unit vector along $\mathbf{v}_{\mathrm{CM}}$. We can calculate that

$$\mathbf{v}_{\mathrm{CM}} = \frac{M}{M+m}(\mathbf{v} - \mathbf{V}) = \frac{M}{M+m}\mathbf{v}_{\mathrm{rel}}$$

where $\mathbf{v}_{\mathrm{rel}}$ is the relative velocity.

After a collision, the CM velocity becomes $\mathbf{v}' = \mathbf{v}'_{\mathrm{CM}} + \mathbf{U}_{\mathrm{CM}}$, where

$$\mathbf{v}'_{\mathrm{CM}} = v_{\mathrm{CM}}\hat{n}'$$

so that the elastic scattering in CM RF frame occurs from direction $\hat{n}$ into direction $\hat{n}'$.

The velocity change is the same in all RF and is equal to

$$\Delta\mathbf{v} = \Delta\mathbf{v}_{\mathrm{CM}} = v_{\mathrm{CM}}(\hat{n}' - \hat{n}) = \frac{M}{M+m}v_{\mathrm{rel}}(\hat{n}' - \hat{n})$$

The averages $\langle \Delta\mathbf{v} \rangle_\Omega$ and $\langle \Delta\mathbf{v}\Delta\mathbf{v}^T \rangle_\Omega$ over the scattering angle (in CM RF) are easily obtained using (5–6). Then we still have to average over $\mathbf{V}$. Now, since the scattering centers are moving, the collision frequency ν depends on

$\mathbf{V}$, because $\nu = \nu(v_{\rm rel})$. Therefore, when averaging over $\mathbf{V}$, we must average the products $\langle \nu \Delta\mathbf{v}\rangle$ and $\langle \nu \Delta\mathbf{v}\Delta\mathbf{v}^T\rangle$:

$$\begin{aligned} \langle \nu\Delta\mathbf{v}\rangle &= -\langle \nu\langle 1-\cos\theta\rangle_\Omega \mathbf{v}_{\rm CM}\rangle_{\mathbf{V}} = -\langle \nu_m \mathbf{v}_{\rm CM}\rangle_{\mathbf{V}} & (12)\\ \langle \nu\Delta\mathbf{v}\Delta\mathbf{v}^T\rangle &= \left\langle \nu v_{\rm CM}^2 \left[\frac{1}{2}\langle \sin^2\theta\rangle_\Omega \overset{\leftrightarrow}{\mathsf{I}}_\perp + \langle (1-\cos\theta)^2\rangle_\Omega \hat{n}\hat{n}^T\right]\right\rangle_{\mathbf{V}} & (13)\end{aligned}$$

where $\overset{\leftrightarrow}{\mathsf{I}}_\perp = \overset{\leftrightarrow}{\mathsf{I}} - \hat{n}\hat{n}^T$ is the perpendicular projection operator and $\nu_m = \nu\langle 1-\cos\theta\rangle_\Omega$ is the momentum transfer rate (which was also discussed earlier in this tutorial). Note that

$$\langle \nu(\Delta\mathbf{v})^2\rangle = \mathrm{Tr}\,\langle \nu\Delta\mathbf{v}\Delta\mathbf{v}^T\rangle = 2\langle \nu_m(v_{\rm rel}) v_{\rm CM}^2\rangle_{\mathbf{V}}$$

4.1 Heavy scatterers

Equations (12–13) are rather useless in general case because $\Delta\mathbf{v}$ is not small and FP equation cannot be used. Let us consider a limiting case of heavy scattering centers, $M \gg m$. Then, although $\Delta\mathbf{v}$ is not small in general case small changes, at least for $v = |\mathbf{v}|$ we can use the FP approach. To get $\langle \nu\Delta v\rangle$, we have to expand $v' = |\mathbf{v}'|$ in series over the small parameter $\alpha = \sqrt{m/M}$. From the energy equipartition theorem, $mv^2 \sim MV^2$, and therefore $V/v \sim \alpha$ and $V/v' \sim \alpha$. We use the fact that $|\mathbf{v}'_{\rm CM}| = |\mathbf{v}_{\rm CM}|$, or $(\mathbf{v}' - \mathbf{U}_{\rm CM})^2 = (\mathbf{v} - \mathbf{U}_{\rm CM})^2$. Multiplying both sides by $(M+m)/(Mv)$, we get:

$$\begin{aligned} \left(\left(1+\frac{m}{M}\right)\frac{\mathbf{v}'}{v} - \left(\frac{m}{M}\hat{v} + \frac{\mathbf{V}}{v}\right)\right)^2 &= \left(\left(1+\frac{m}{M}\right)\hat{v} - \left(\frac{m}{M}\hat{v} + \frac{\mathbf{V}}{v}\right)\right)^2 \\ \left(\frac{\mathbf{v}'}{v} + \frac{m}{M}\left(\frac{\mathbf{v}'}{v} - \hat{v}\right) - \frac{\mathbf{V}}{v}\right)^2 &= \left(\hat{v} - \frac{\mathbf{V}}{v}\right)^2 \end{aligned}$$

Now, carefully expand in series over α, keeping only terms up to $O(\alpha^2)$:

$$\frac{v'^2}{v^2} + 2\frac{m}{M}\hat{n}'\cdot(\hat{n}'-\hat{n}) - 2\frac{\mathbf{v}'}{v}\cdot\frac{\mathbf{V}}{v} = 1 - 2\hat{v}\cdot\frac{\mathbf{V}}{v}$$

$$\frac{\Delta(v^2)}{v^2} = \frac{v'^2}{v^2} - 1 = -2\frac{m}{M}(1-\hat{n}'\cdot\hat{n}) + 2\frac{(\mathbf{v}'-\mathbf{v})}{v}\cdot\frac{\mathbf{V}}{v}$$

We use $\mathbf{v}' - \mathbf{v} = \mathbf{v}'_{\rm CM} - \mathbf{v}_{\rm CM} = v_{\rm CM}(\hat{n}' - \hat{n})$ and

$$v_{\rm CM} = v_{\rm rel} + O(\alpha^2 v) = v - \hat{n}\cdot\mathbf{V} + O(\alpha^2 v) \tag{14}$$

and get

$$\frac{\Delta(v^2)}{v^2} = -2\frac{m}{M}(1-\hat{n}'\cdot\hat{n}) + 2\left(\left(\hat{n}\cdot\frac{\mathbf{V}}{v}\right)^2 - \left(\hat{n}\cdot\frac{\mathbf{V}}{v}\right)\left(\hat{n}'\cdot\frac{\mathbf{V}}{v}\right)\right) - 2(\hat{n}-\hat{n}')\cdot\frac{\mathbf{V}}{v} \tag{15}$$

where the first two terms on the RHS are $O(\alpha^2)$ and the last is $O(\alpha)$. Let us average over the scattering direction. We use $\hat{n}'\cdot\hat{n} = \cos\theta$ and $\langle\hat{n}'\rangle_\Omega = \hat{n}\langle\cos\theta\rangle_\Omega$:

$$\frac{\langle\Delta(v^2)\rangle_\Omega}{v^2} = 2\langle 1-\cos\theta\rangle_\Omega\left(-\frac{m}{M} + \left(\hat{n}\cdot\frac{\mathbf{V}}{v}\right)^2 - \hat{n}\cdot\frac{\mathbf{V}}{v}\right)$$

Before we average over $\mathbf{V}$, let us switch from $\hat{n}$ to $\hat{v}$, using

$$\hat{n} = \left(\hat{v} - \frac{\mathbf{V}}{v}\right)\left|\hat{v} - \frac{\mathbf{V}}{v}\right|^{-1}$$

To order $O(\alpha)$,

$$\hat{n} = \hat{v}\left(1 + \hat{v}\cdot\frac{\mathbf{V}}{v}\right) - \frac{\mathbf{V}}{v} = \hat{v} - \frac{\mathbf{V}_\perp}{v}$$

where $\mathbf{V}_\perp = \mathbf{V} - \hat{v}(\hat{v}\cdot\mathbf{V})$. Thus, to the same order of $O(\alpha^2)$,

$$\frac{\langle\Delta(v^2)\rangle_\Omega}{v^2} = 2\langle 1-\cos\theta\rangle_\Omega\left(-\frac{m}{M} + \left(\hat{v}\cdot\frac{\mathbf{V}}{v}\right)^2 - \hat{v}\cdot\frac{\mathbf{V}}{v} + \frac{\mathbf{V}_\perp^2}{v^2}\right)$$

Now it is time to average over $\mathbf{V}$:

$$\frac{\langle\nu\Delta(v^2)\rangle}{v^2} = 2\left\langle\nu_m(v_{\rm rel})\left(-\frac{m}{M} + \left(\hat{v}\cdot\frac{\mathbf{V}}{v}\right)^2 - \hat{v}\cdot\frac{\mathbf{V}}{v} + \frac{\mathbf{V}_\perp^2}{v^2}\right)\right\rangle_{\mathbf{V}}$$

We use (14) to expand $\nu_m(v_{\rm rel}) = \nu_m(v) - (\hat{v}\cdot\mathbf{V})\nu'_m(v) + O(\alpha^2\nu)$. Substituting and keeping terms up to $O(\alpha^2\nu)$, we get

$$\begin{aligned}\frac{\langle\nu\Delta(v^2)\rangle}{v^2} &= -2\frac{m}{M}\nu_m + 2\nu_m\frac{\langle\mathbf{V}_\perp^2\rangle}{v^2} + 2\left\langle\left(\hat{v}\cdot\frac{\mathbf{V}}{v}\right)^2\right\rangle_{\mathbf{V}}(\nu_m + v\nu'_m)\\ &= -2\frac{m}{M}\nu_m + \frac{2}{3}\frac{\langle\mathbf{V}^2\rangle}{v^2}(3\nu_m + v\nu'_m)\end{aligned}$$

where we used $\langle \mathbf{V}_\perp^2 \rangle = \frac{2}{3}\langle \mathbf{V}^2 \rangle$ and $\langle (\hat{v}\cdot\mathbf{V})^2 \rangle = \frac{1}{3}\langle \mathbf{V}^2 \rangle$ since $\mathbf{V}$ is isotropic. Note that the term of order $O(\alpha)$ after averaging became $O(\alpha^2)$.

To write FP equation for the energy $\mathcal{E} = \frac{mv^2}{2}$, beside $\langle \nu\Delta\mathcal{E} \rangle = \mathcal{E}\langle \nu\Delta(v^2) \rangle$ we also need $\langle \nu(\Delta\mathcal{E})^2 \rangle$. Using only the last (biggest) term in (15), we get:

$$\frac{(\Delta\mathcal{E})^2}{\mathcal{E}^2} = \left((\hat{n}-\hat{n}')\cdot\frac{\mathbf{V}}{v}\right)^2 + O(\alpha^3)$$

It is more convenient to average first over $\mathbf{V}$, keeping $\hat{n}$ and $\hat{n}'$ constant. We can do it here, unlike the calculation for $\Delta(v^2)$ because our error in doing this (due to the fact that V is not completely isotropic relative to $\hat{n}$ and $\hat{n}'$ in the CM RF) will be of $O(\alpha^3)$. So

$$\frac{\langle (\Delta\mathcal{E})^2 \rangle_{\mathbf{V}}}{\mathcal{E}^2} = \frac{4}{3}\left|\hat{n}-\hat{n}'\right|^2 \frac{\langle \mathbf{V}^2 \rangle}{v^2}$$

Finally, averaging over the scattering direction,

$$\frac{\langle \nu(\Delta\mathcal{E})^2 \rangle}{\mathcal{E}^2} = \frac{8}{3}\nu_m\frac{\langle \mathbf{V}^2 \rangle}{v^2}$$

Summarizing for a thermal gas of heavy particles, $\langle \mathbf{V}^2 \rangle = 3T/M$:

$$\begin{aligned}
\langle \nu\Delta\mathcal{E} \rangle &= -2\frac{m}{M}\nu_m\mathcal{E} + \frac{m}{M}\left(3\nu_m + 2\mathcal{E}\frac{d\nu_m}{d\mathcal{E}}\right)T \\
\langle \nu(\Delta\mathcal{E})^2 \rangle &= 4\frac{m}{M}\nu_m T\mathcal{E}
\end{aligned}$$

where we switched from $\nu_m' = d\nu_m/dv$ to derivative over $\mathcal{E}$. The FP equation in the form (3) is

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial\mathcal{E}}\left(Fn - D\frac{\partial n}{\partial\mathcal{E}}\right)$$

where

$$\begin{aligned}
F(\mathcal{E}) &= \langle \nu\Delta\mathcal{E} \rangle - \frac{1}{2}\frac{\partial\langle \nu(\Delta\mathcal{E})^2 \rangle}{\partial\mathcal{E}} = -2\frac{m}{M}\nu_m\left(\mathcal{E} - \frac{T}{2}\right) \\
D(\mathcal{E}) &= \frac{1}{2}\langle \nu(\Delta\mathcal{E})^2 \rangle = 2\frac{m}{M}\nu_m T\mathcal{E}
\end{aligned}$$

Note that in an n-dimensional gas there is a requirement that

$$\frac{F(\mathcal{E})}{D(\mathcal{E})} = -\frac{1}{T} + \frac{n-2}{2\mathcal{E}}$$

in order to get a stationary Maxwellian distribution $n(\mathcal{E}) \propto \mathcal{E}^{(n-2)/2}e^{-\mathcal{E}/T}$.

References

[1] D. R. Nicholson, “Introduction to Plasma theory”, Appendix B, J. Wiley and Sons, New York (1983).