Fourier transform (FT) properties

NGL

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Abstract

This is a summary of basic formulas using Fourier transforms, with conventions accepted in physics.

1 Conventions

All integrals and sums are from $-\infty$ to $+\infty$, if not noted otherwise. The "physics" convention (complex signals $\propto e^{-i\omega t}$) is used. The "engineering" convention (complex signals $\propto e^{j\omega t}$) is obtained by the transformation $i \to -j$. The field theory convention (factor of $1/2\pi$ is in the frequency/wave-vector domain integrals) is used. The Fourier transformed functions are denoted with a tilde.

2 Important formulas

$$\int e^{i\omega t}\, dt = 2\pi\delta(\omega) \tag{1}$$

$$\int e^{-i\omega t}\, \frac{d\omega}{2\pi} = \delta(t) \tag{2}$$

$$\sum_n e^{2\pi i x n} = \sum_m \delta(x-m) \tag{3}$$

3 Continuous Fourier transform (CFT)

$$\tilde{X}(\omega) = \int X(t) e^{i\omega t}\, dt \quad \Longleftrightarrow \quad X(t) = \int \tilde{X}(\omega) e^{-i\omega t}\, \frac{d\omega}{2\pi} \tag{4}$$

Time/frequency inversion, conjugation:

$$\begin{array}{rcll} X^*(t) = X(t) & \Longleftrightarrow & \tilde{X}^*(\omega) = \tilde{X}(-\omega) & \text{(real signals)} \\ Y(t) = X(-t) & \Longleftrightarrow & \tilde{Y}(\omega) = \tilde{X}(-\omega) & \\ Y(t) = X^*(t) & \Longleftrightarrow & \tilde{Y}(\omega) = \tilde{X}^*(-\omega) & \end{array}$$

Convolution of complex signals (filtering) in time and frequency domains:

$$Z(t) = \int X(t')Y(t-t')\, dt' \quad \Longleftrightarrow \quad \tilde{Z}(\omega) = \tilde{X}(\omega)\tilde{Y}(\omega)$$

$$Z(t) = X(t)Y(t) \quad \Longleftrightarrow \quad \tilde{Z}(\omega) = \int \tilde{X}(\omega')\tilde{Y}(\omega-\omega')\, \frac{d\omega'}{2\pi}$$

Time shift is obtained by substituting $Y(t) = \delta(t - \Delta t)$, $\tilde{Y}(\omega) = e^{i\omega\Delta t}$:

$$Z(t) = X(t - \Delta t) \quad \Longleftrightarrow \quad \tilde{Z}(\omega) = e^{i\omega\Delta t}\tilde{X}(\omega)$$

Frequency shift $\tilde{Y}(\omega) = 2\pi\delta(\omega - \Delta\omega)$, $Y(t) = e^{-i\Delta\omega t}$:

$$Z(t) = e^{-i\Delta\omega t}X(t) \quad \Longleftrightarrow \quad \tilde{Z}(\omega) = \tilde{X}(\omega - \Delta\omega)$$

For the space/wave-vector integrals the formulas are analogous:

$$\begin{aligned} \tilde{X}(\omega, \mathbf{k}) &= \iint X(t, \mathbf{r}) e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}\, dt\, d^3\mathbf{r} \\ X(t, \mathbf{r}) &= \iint \tilde{X}(\omega, \mathbf{k}) e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}}\, \frac{d\omega}{2\pi}\frac{d^3\mathbf{k}}{(2\pi)^3} \end{aligned}$$

etc., where integrals are from $-\infty$ to $+\infty$.

Differentiation, multiplication:

time-space domain	$\leftrightarrow$	frequency-wavevector domain
$\frac{\partial}{\partial t}$	$\leftrightarrow$	$-i\omega$
t	$\leftrightarrow$	$-i\frac{\partial}{\partial\omega}$
∇	$\leftrightarrow$	$i\mathbf{k}$
$\mathbf{r}$	$\leftrightarrow$	$i\nabla_{\mathbf{k}}$

3.1 Energy and power spectra

Power is defined as $|X(t)|^2$. The energy is power intergrated over time. The **energy spectrum** of a finite signal $X(t)$ is energy contained in frequency interval $df = d\omega/(2\pi)$ and is equal to $E(\omega) = \left|\tilde{X}(\omega)\right|^2$:

$$\int |X(t)|^2 \, dt = \int E(\omega) \, \frac{d\omega}{2\pi} = \int \left|\tilde{X}(\omega)\right|^2 \, \frac{d\omega}{2\pi}$$

Note that the integration is from $-\infty$ to $+\infty$. Usually in engineering the spectrum is only defined for positive frequencies, so the experimental spectrum will be $E_{\text{exp}}(\omega) = 2\left|\tilde{X}(\omega)\right|^2$ in this case. See also equation (11) below.

For stationary (ergodic) complex random signals we must use a **power spectrum** $S(\omega)$ instead. It is related to the **correlation function** $K(\tau)$, and they have the following properties:

$$K(\tau) = \langle X^*(t)X(t+\tau)\rangle \qquad \text{(definition)} \tag{5}$$

$$K(-\tau) = K^*(\tau) \qquad \text{(or symmetric for real signals)} \tag{6}$$

$$S(\omega) = \tilde{K}(\omega) \qquad \text{(Wiener-Khinchine relation)} \tag{7}$$

$$S(\omega) = S^*(\omega) \qquad \text{(spectrum is always real)} \tag{8}$$

$$\langle \tilde{X}^*(\omega_1)\tilde{X}(\omega_2)\rangle = 2\pi\delta(\omega_1 - \omega_2)S(\omega_1) \tag{9}$$

$$\int S(\omega) \, \frac{d\omega}{2\pi} = \langle |X(t)|^2 \rangle \tag{10}$$

Note that $S(\omega)$ is per Hz, since the frequency in cycles per second is $f = \omega/(2\pi)$, as seen from the last formula. For practical (experimental) purposes it is more correct to use the spectrum defined only for positive frequencies:

$$S_{\text{exp}}(f) = 2S(\omega) \tag{11}$$

which has the property

$$\int_0^\infty S(f)\, df = \langle X^2(t) \rangle$$

4 FT of a function on a finite interval (Fourier series)

Consider interval $t \in [0, T]$. Then the FT of a function $X(t)$ can be defined as a discrete infinite set of numbers $\tilde{X}_k$:

$$\tilde{X}_k = \int_0^T X(t) e^{2\pi i k t/T}\, dt \quad \Longleftrightarrow \quad X(t) = \frac{1}{T} \sum_k \tilde{X}_k e^{-2\pi i k t/T} \tag{12}$$

These equations are derived using equation (3). They can be set in a form that is more similar to the continuous formulas (4) using $\Delta\omega = 2\pi/T$ and $\omega_k = 2\pi k/T = k\Delta\omega$:

$$\tilde{X}_k = \int_0^T X(t) e^{i\omega_k t}\, dt \quad \Longleftrightarrow \quad X(t) = \sum_k \tilde{X}_k e^{-i\omega_k t} \frac{\Delta\omega}{2\pi} \tag{13}$$

FT of a periodic function $X(t+T) = X(t)$ is

$$\tilde{X}(\omega) = \sum_k \tilde{X}_k\, \Delta\omega\, \delta(\omega - \omega_k) = \sum_k \tilde{X}_k \delta\left(\frac{\omega}{\Delta\omega} - k\right) \tag{14}$$

5 FT on a discrete infinite set

Consider a set of points $n = -\infty \ldots + \infty$, with a function X_n defined on them. A common situation is, e.g., a crystallic grid. FT can be defined as a function $\tilde{X}(\omega)$ taken on a finite interval $[0, 2\pi]$:

$$\tilde{X}(\omega) = \sum_n X_n e^{i\omega n} \quad \Longleftrightarrow \quad X_n = \int_0^{2\pi} \tilde{X}(\omega) e^{-i\omega n} \frac{d\omega}{2\pi} \tag{15}$$

If the distance between discrete points is Δt, then we can introduce $t_n = n\Delta t$, and $\omega_{\max} = 2\pi/\Delta t$:

$$\tilde{X}(\omega) = \sum_n X_n e^{i\omega t_n} \Delta t \quad \Longleftrightarrow \quad X_n = \int_0^{\omega_{\max}} \tilde{X}(\omega) e^{-i\omega t_n} \frac{d\omega}{2\pi} \tag{16}$$

6 Discrete Fourier transform (DFT)

DFT is taken over N points. The "engineering" convention is used here because it is used in MATLAB. All indeces start with 1 (FORTRAN/MATLAB convention). If the index is outside interval $\{1, N\}$, then mod N is assumed (cyclic signals $X_{n+N} \equiv X_n$).

$$\tilde{X}_k = \sum_{n=1}^{N} X_n e^{-\frac{2\pi j}{N}(n-1)(k-1)} \qquad X_n = \frac{1}{N}\sum_{k=1}^{N} \tilde{X}_n e^{\frac{2\pi j}{N}(n-1)(k-1)}$$

For real signals we have

$$\tilde{X}^*_{N+2-k} = \tilde{X}_k$$

For complex signals

$$Y_n = X^*_{N+2-n} \quad \Longleftrightarrow \quad \tilde{Y}_k = \tilde{X}^*_k$$
$$Y_n = X^*_n \quad \Longleftrightarrow \quad \tilde{Y}_k = \tilde{X}^*_{N+2-k}$$

Correlation for cyclic random ergodic signals:

$$K_m = \langle X_n X_{n+m-1} \rangle$$

is independent of n and has properties analogous to the CFT case:

$$\langle \tilde{X}^*_k \tilde{X}_l \rangle = \delta_{kl} N \tilde{K}_k$$

and the WK relation

$$S_k = \tilde{K}_k, \qquad K_1 = \langle X_n^2 \rangle = \frac{1}{N}\sum_{k=1}^{N} S_k \tag{17}$$

The convolution (filtering) of complex signals:

$$Z_n = \sum_{m=1}^{N} X_m Y_{n-m+1} \quad \Longleftrightarrow \quad \tilde{Z}_k = \tilde{X}_k \tilde{Y}_k$$
$$Z_n = \sum_{m=1}^{N} X_m Y^*_{m-n+1} \quad \Longleftrightarrow \quad \tilde{Z}_k = \tilde{X}_k \tilde{Y}^*_k$$

Time shift is obtained using $Y_n = \delta_{n,s+1}$, $\tilde{Y}_k = e^{-\frac{2\pi j}{N}s(k-1)}$:

$$Z_n = X_{n-s} \quad \Longleftrightarrow \quad \tilde{Z}_k = e^{-\frac{2\pi j}{N}s(k-1)} \tilde{X}_k$$

6.1 Index starting at zero, $i \leftrightarrow -j$

Just for reference (now $k = 0 \dots N-1$, $n = 0 \dots N-1$):

$$\tilde{X}_k = \sum_{n=0}^{N-1} X_n e^{2\pi ink/N} \quad \Longleftrightarrow \quad X_n = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}_n e^{-2\pi ink/N}$$

$$X \in \mathbb{R} \rightarrow \tilde{X}^*_{N-k} = \tilde{X}_k$$

$$K_m = \langle X_n X_{n+m} \rangle$$

K_0 is the power. The convolution:

$$\begin{aligned} Z_n = \sum_{m=0}^{N-1} X_m Y_{n-m} \quad &\Longleftrightarrow \quad \tilde{Z}_k = \tilde{X}_k \tilde{Y}_k \\ Z_n = \sum_{m=0}^{N-1} X_m Y^*_{m-n} \quad &\Longleftrightarrow \quad \tilde{Z}_k = \tilde{X}_k \tilde{Y}^*_k \end{aligned}$$

Time shift ($Y_n = \delta_{n,s}$, $\tilde{Y}_k = e^{2\pi isk/N}$):

$$Z_n = X_{n-s} \quad \Longleftrightarrow \quad \tilde{Z}_k = e^{2\pi isk/N} \tilde{X}_k$$

6.2 Spectrum of a discretized continuous signal

The spectrum of discretized signal with time interval Δt is obtained in the following way. The times and frequencies are

$$\begin{aligned} t_n &= (n-1)\Delta t \\ T &= t_{N+1} = N\Delta t \\ f_k &= (k-1)\Delta f \\ \Delta f &= \frac{1}{T} = \frac{F}{N} \\ F &= f_{N+1} = N\Delta f = \frac{1}{\Delta t} \end{aligned}$$

where T is the total time sample, or the period of the cyclic signal, F is the discretization frequency, Δf is the frequency resolution. The Nyquist frequency is the maximum frequency for which the spectrum can be obtained:

$$f_{\text{Nyquist}} = \frac{F}{2} = \frac{1}{2\Delta t}$$

The spectrum per Hz is determined by comparing (7) and (17):

$$S(\omega_k)\Delta f = \frac{1}{N} S_k$$

from where

$$S(\omega_k) = \frac{S_k}{F} = S_k \Delta t = \tilde{K}_k \Delta t$$

The experimental spectrum (11):

$$S_{\text{exp}}(f_k) = 2S(\omega_k) = 2\tilde{K}_k \Delta t$$