Fourier transform (FT) properties

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Abstract

This is a summary of basic formulas using Fourier transforms, with conventions accepted in physics.

1 Conventions

All integrals and sums are from $-\infty$ to $+\infty$, if not noted otherwise. The “physics” convention (complex signals $\propto e^{-i\omega t}$) is used. The “engineering” convention (complex signals $\propto e^{j\omega t}$) is obtained by the transformation $i \rightarrow -j$. The field theory convention (factor of $1/2\pi$ is in the frequency/wave-vector domain integrals) is used. The Fourier transformed functions are denoted with a tilde.

2 Important formulas

\[ \int e^{i\omega t} \, dt = 2\pi \delta(\omega) \quad (1) \]
\[ \int e^{-i\omega t} \frac{d\omega}{2\pi} = \delta(t) \quad (2) \]
\[ \sum_n e^{2\pi i xn} = \sum_m \delta(x - m) \quad (3) \]
3 Continuous Fourier transform (CFT)

\[ \tilde{X}(\omega) = \int X(t)e^{i\omega t} \, dt \quad \iff \quad X(t) = \int \tilde{X}(\omega)e^{-i\omega t} \frac{d\omega}{2\pi} \]  \hspace{1cm} (4)

Time/frequency inversion, conjugation:

\[ X^*(t) = X(t) \quad \iff \quad \tilde{X}^*(\omega) = \tilde{X}(-\omega) \quad \text{(real signals)} \]

\[ Y(t) = X(-t) \quad \iff \quad \tilde{Y}(\omega) = \tilde{X}(\omega) \]

\[ Y(t) = X^*(t) \quad \iff \quad \tilde{Y}(\omega) = \tilde{X}^*(-\omega) \]

Convolution of complex signals (filtering) in time and frequency domains:

\[ Z(t) = \int X(t')Y(t-t') \, dt' \quad \iff \quad \tilde{Z}(\omega) = \tilde{X}(\omega)\tilde{Y}(\omega) \]

\[ Z(t) = X(t)Y(t) \quad \iff \quad \tilde{Z}(\omega) = \int \tilde{X}(\omega')\tilde{Y}(\omega-\omega') \frac{d\omega'}{2\pi} \]

Time shift is obtained by substituting \( Y(t) = \delta(t-\Delta t), \tilde{Y}(\omega) = e^{i\omega \Delta t} \):

\[ Z(t) = X(t-\Delta t) \quad \iff \quad \tilde{Z}(\omega) = e^{i\omega \Delta t}\tilde{X}(\omega) \]

Frequency shift \( \tilde{Y}(\omega) = 2\pi \delta(\omega-\Delta \omega), Y(t) = e^{-i\Delta \omega t} \):

\[ Z(t) = e^{-i\Delta \omega t}X(t) \quad \iff \quad \tilde{Z}(\omega) = \tilde{X}(\omega-\Delta \omega) \]

For the space/wave-vector integrals the formulas are analogous:

\[ \tilde{X}(\omega, k) = \iiint X(t, r)e^{i\omega t - ik \cdot r} \, dt \, d^3r \]

\[ X(t, r) = \iiint \tilde{X}(\omega, k)e^{-i\omega t + ik \cdot r} \frac{d\omega}{2\pi} \frac{d^3k}{(2\pi)^3} \]

etc., where integrals are from \(-\infty\) to \(+\infty\).

Differentiation, multiplication:

\[ \frac{\partial}{\partial t} \iff -i\omega \]

\[ t \iff -i \frac{\partial}{\partial \omega} \]

\[ \nabla \iff ik \]

\[ r \iff i\nabla_k \]
3.1 Energy and power spectra

Power is defined as $|X(t)|^2$. The energy is power integrated over time. The energy spectrum of a finite signal $X(t)$ is energy contained in frequency interval $df = d\omega/(2\pi)$ and is equal to $E(\omega) = |\tilde{X}(\omega)|^2$:

$$
\int |X(t)|^2 \, dt = \int E(\omega) \frac{d\omega}{2\pi} = \int |\tilde{X}(\omega)|^2 \frac{d\omega}{2\pi}
$$

Note that the integration is from $-\infty$ to $+\infty$. Usually in engineering the spectrum is only defined for positive frequencies, so the experimental spectrum will be $E_{\text{exp}}(\omega) = 2|\tilde{X}(\omega)|^2$ in this case. See also equation (11) below.

For stationary (ergodic) complex random signals we must use a power spectrum $S(\omega)$ instead. It is related to the correlation function $K(\tau)$, and they have the following properties:

$$
K(\tau) = \langle X^*(t)X(t+\tau) \rangle \quad \text{(definition)} \quad (5)
$$

$$
K(-\tau) = K^*(\tau) \quad \text{(or symmetric for real signals)} \quad (6)
$$

$$
S(\omega) = \tilde{K}(\omega) \quad \text{(Wiener-Khinchine relation)} \quad (7)
$$

$$
S(\omega) = S^*(\omega) \quad \text{(spectrum is always real)} \quad (8)
$$

$$
\langle \tilde{X}^*(\omega_1)\tilde{X}(\omega_2) \rangle = 2\pi \delta(\omega_1 - \omega_2)S(\omega_1) \quad (9)
$$

$$
\int S(\omega) \frac{d\omega}{2\pi} = \langle |X(t)|^2 \rangle \quad (10)
$$

Note that $S(\omega)$ is per Hz, since the frequency in cycles per second is $f = \omega/(2\pi)$, as seen from the last formula. For practical (experimental) purposes it is more correct to use the spectrum defined only for positive frequencies:

$$
S_{\text{exp}}(f) = 2S(\omega) \quad (11)
$$

which has the property

$$
\int_0^\infty S(f) \, df = \langle X^2(t) \rangle
$$
4 FT of a function on a finite interval (Fourier series)

Consider interval $t \in [0, T]$. Then the FT of a function $X(t)$ can be defined as a discrete infinite set of numbers $\tilde{X}_k$:

$$\tilde{X}_k = \int_0^T X(t)e^{2\pi ikt/T} \, dt \iff X(t) = \frac{1}{T} \sum_k \tilde{X}_k e^{-2\pi ikt/T} \quad (12)$$

These equations are derived using equation (3). They can be set in a form that is more similar to the continuous formulas (4) using $\Delta \omega = 2\pi/T$ and $\omega_k = 2\pi k/T = k\Delta \omega$:

$$\tilde{X}_k = \int_0^T X(t)e^{i\omega_k t} \, dt \iff X(t) = \sum_k \tilde{X}_k e^{-i\omega_k t} \frac{\Delta \omega}{2\pi} \quad (13)$$

FT of a periodic function $X(t + T) = X(t)$ is

$$\tilde{X}(\omega) = \sum_k \tilde{X}_k \Delta \omega \delta(\omega - \omega_k) = \sum_k \tilde{X}_k \delta \left( \frac{\omega}{\Delta \omega} - k \right) \quad (14)$$

5 FT on a discrete infinite set

Consider a set of points $n = -\infty \ldots + \infty$, with a function $X_n$ defined on them. A common situation is, e.g., a crystallic grid. FT can be defined as a function $\tilde{X}(\omega)$ taken on a finite interval $[0, 2\pi]$:

$$\tilde{X}(\omega) = \sum_n X_n e^{i\omega n} \iff X_n = \int_0^{2\pi} \tilde{X}(\omega)e^{-i\omega n} \frac{d\omega}{2\pi} \quad (15)$$

If the distance between discrete points is $\Delta t$, then we can introduce $t_n = n\Delta t$, and $\omega_{\text{max}} = 2\pi/\Delta t$:

$$\tilde{X}(\omega) = \sum_n X_n e^{i\omega t_n} \Delta t \iff X_n = \int_0^{\omega_{\text{max}}} \tilde{X}(\omega)e^{-i\omega t_n} \frac{d\omega}{2\pi} \quad (16)$$
6 Discrete Fourier transform (DFT)

DFT is taken over \( N \) points. The “engineering” convention is used here because it is used in MATLAB. All indices start with 1 (FORTRAN/MATLAB convention). If the index is outside interval \{1, \( N \}\}, then \( \text{mod} \ N \) is assumed (cyclic signals \( X_{n+N} \equiv X_n \)).

\[
\tilde{X}_k = \sum_{n=1}^{N} X_ne^{-\frac{2\pi j}{N}(n-1)(k-1)} \quad X_n = \frac{1}{N} \sum_{k=1}^{N} \tilde{X}_ne^{\frac{2\pi j}{N}(n-1)(k-1)}
\]

For real signals we have

\[ \tilde{X}_{N+2-k} = \tilde{X}_k \]

For complex signals

\[
Y_n = X_{N+2-n}^* \iff \tilde{Y}_k = \tilde{X}_k^* \\
Y_n^* = X_n^* \iff \tilde{Y}_k^* = \tilde{X}_k^*_{N+2-k}
\]

Correlation for cyclic random ergodic signals:

\[ K_m = \langle X_nX_{n+m-1} \rangle \]

is independent of \( n \) and has properties analogous to the CFT case:

\[ \langle \tilde{X}_k^* \tilde{X}_l \rangle = \delta_{kl}N\tilde{K}_k \]

and the WK relation

\[
S_k = \tilde{K}_k, \quad K_1 = \langle X_n^2 \rangle = \frac{1}{N} \sum_{k=1}^{N} S_k \tag{17}
\]

The convolution (filtering) of complex signals:

\[
Z_n = \sum_{m=1}^{N} X_m Y_{n-m+1} \iff \tilde{Z}_k = \tilde{X}_k\tilde{Y}_k \\
Z_n = \sum_{m=1}^{N} X_m Y_{n-m+1}^* \iff \tilde{Z}_k = \tilde{X}_k\tilde{Y}_k^*
\]

Time shift is obtained using \( Y_n = \delta_{n,s+1}, \tilde{Y}_k = e^{-\frac{2\pi j s}{N}(k-1)} \):

\[
Z_n = X_{n-s} \iff \tilde{Z}_k = e^{-\frac{2\pi j s}{N}(k-1)}\tilde{X}_k
\]
6.1 Index starting at zero, \( i \leftrightarrow -j \)

Just for reference (now \( k = 0 \ldots N - 1 \), \( n = 0 \ldots N - 1 \)):

\[
\tilde{X}_k = \sum_{n=0}^{N-1} X_n e^{2\pi i nk/N} \quad \iff \quad X_n = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}_k e^{-2\pi i nk/N}
\]

\( X \in \mathbb{R} \rightarrow \tilde{X}^*_{N-k} = \tilde{X}_k \)

\( K_m = \langle X_n X_{n+m} \rangle \)

\( K_0 \) is the power. The convolution:

\[
Z_n = \sum_{m=0}^{N-1} X_m Y_{n-m} \quad \iff \quad \tilde{Z}_k = \tilde{X}_k \tilde{Y}_k
\]

\[
Z_n = \sum_{m=0}^{N-1} X_m Y^*_{m-n} \quad \iff \quad \tilde{Z}_k = \tilde{X}_k \tilde{Y}^*_k
\]

Time shift (\( Y_n = \delta_{n,s}, \tilde{Y}_k = e^{2\pi isk/N} \)):

\[
Z_n = X_{n-s} \quad \iff \quad \tilde{Z}_k = e^{2\pi isk/N} \tilde{X}_k
\]

6.2 Spectrum of a discretized continuous signal

The spectrum of discretized signal with time interval \( \Delta t \) is obtained in the following way. The times and frequencies are

\[
t_n = (n-1)\Delta t \\
T = t_{N+1} = N\Delta t \\
f_k = (k-1)\Delta f \\
\Delta f = \frac{1}{T} = \frac{F}{N} \\
F = f_{N+1} = N\Delta f = \frac{1}{\Delta t}
\]
where $T$ is the total time sample, or the period of the cyclic signal, $F$ is the discretization frequency, $\Delta f$ is the frequency resolution. The Nyquist frequency is the maximum frequency for which the spectrum can be obtained:

$$f_{\text{Nyquist}} = \frac{F}{2} = \frac{1}{2\Delta t}$$

The spectrum per Hz is determined by comparing (7) and (17):

$$S(\omega_k) \Delta f = \frac{1}{N} S_k$$

from where

$$S(\omega_k) = \frac{S_k}{F} = S_k \Delta t = \tilde{K}_k \Delta t$$

The experimental spectrum (11):

$$S_{\text{exp}}(f_k) = 2S(\omega_k) = 2\tilde{K}_k \Delta t$$