Error functions

NGL

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We use notations of [*Abramowitz and Stegun, 1965*, ch. 7], where possible.

1 Error function

Error function (Gauss error function) is defined as

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{1}$$

has properties $\operatorname{erf}(-\infty) = -1$, $\operatorname{erf}(+\infty) = 1$, $\operatorname{erf}(-x) = -\operatorname{erf}(x)$. Complementary error function is defined as

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt = 1 - \operatorname{erf} x \tag{2}$$

2 w function

This function does not have a name in [*Abramowitz and Stegun, 1965*, ch. 7]. However, other authors give it the following names:

- Kramp function [*Mikhailovskiy, 1975*, p. 36];
- error function of a complex argument [*Baumjohann and Treumann, 1997*, p. 310]
- probability integral of a complex argument [*Fadeeva and Terent'ev, 1954*].

- Fadeeva (Faddeeva) function [e.g., Wikipedia], because it was tabulated by [*Fadeeva and Terent'ev, 1954*]

It is defined as

$$w(x) = e^{-x^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^x e^{t^2}\,dt\right) = e^{-x^2}[1 + \mathrm{erf}\,(ix)] = e^{-x^2}\mathrm{erfc}\,(-ix) \quad (3)$$

Integral representations:

$$w(x) = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}\,dt}{x-t} = \frac{2ix}{\pi}\int_0^{\infty}\frac{e^{-t^2}\,dt}{x^2-t^2} \quad (4)$$

where $\Im x > 0$. These integral representations can be converted to (3) using

$$\frac{1}{x+i\Delta-t} = -2i\int_0^{\infty} e^{2i(x+i\Delta-t)u}\,du$$

3 Other definitions

The names in parenthesis are introduced by me, after (an) author(s) who used them before.

3.1 Plasma dispersion function $Z(x)$

Most authors in the USA use the *plasma dispersion function* [*Fried and Conte, 1961*]:

$$Z(x) \equiv i\sqrt{\pi}w(x) \quad (5)$$

We see that

$$Z(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{e^{-t^2}}{t-x}\,dt \quad (6)$$

and

$$Z(x) = 2ie^{-x^2}\int_{-\infty}^{ix} e^{-t^2}\,dt = i\sqrt{\pi}e^{-x^2}[1 + \mathrm{erf}\,(ix)] \quad (7)$$

and

$$Z(x) = i\sqrt{\pi}e^{-x^2}\mathrm{erfc}\,(-ix) \quad (8)$$

3.2 (Jackson) function $G(x)$

Another function useful in plasma physics is [*Jackson, 1960*]

$$G(x) = 1 + i\sqrt{\pi}xw(x) = 1 + xZ(x) = -Z'(x)/2 \tag{9}$$

Integral representation

$$G(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{te^{-t^2}\,dt}{t - x} \tag{10}$$

where again $\Im x > 0$.

3.3 Fresnel functions

Fresnel functions are defined as

$$C(x) + iS(x) = \int_0^x e^{\pi i t^2/2}\,dt$$

3.4 Dawson integral

This is introduced by name of Dawson integral in [*Abramowitz and Stegun, 1965*, ch. 7] and used in [*Stix, 1962*, p. 178]:

$$S(z) = e^{-z^2} \int_0^z e^{t^2}\,dt$$

3.5 (Sitenko) function $\varphi(x)$

Another function [*Sitenko, 1982*, p. 24] is φ, defined only for real arguments:

$$\varphi(x) = 2xe^{-x^2} \int_0^x e^{t^2}\,dt \tag{11}$$

so that

$$G(x) = 1 - \varphi(x) + i\sqrt{\pi}xe^{-x^2} \tag{12}$$

3.6 (Fried-Conte) function $Y(x)$

[*Fried and Conte, 1961*] introduces

$$Y(x) = \frac{e^{-x^2}}{x} \int_0^x e^{t^2}\,dt \tag{13}$$

so that for real argument

$$Z(x) = i\sqrt{\pi}e^{-x^2} - 2xY(x) \tag{14}$$

3.7 Alternative definitions.

In [*Mikhailovskiy, 1975*], notation is $Z_{\mathrm{Mikh}}(x) \equiv i\sqrt{\pi}xw(x) = xZ(x)$ (also w is denoted as W_{Mikh}).

4 Asymptotic fomulas for real argument

- $|x| \gg 1$

$$W(x) = \frac{i}{\sqrt{\pi}x}\left(1 + \frac{1}{2x^2} + \frac{3}{4x^4} + \dots\right) + e^{-x^2} \tag{15}$$

$$Z(x) = -\frac{1}{x}\left(1 + \frac{1}{2x^2} + \frac{3}{4x^4} + \dots\right) + i\sqrt{\pi}e^{-x^2} \tag{16}$$

$$G(x) = -\frac{1}{2x^2} - \frac{3}{4x^4} - \dots + i\sqrt{\pi}xe^{-x^2}, \tag{17}$$

- $|x| \ll 1$

$$W(x) = 1 + \frac{2ix}{\sqrt{\pi}} + \dots \tag{18}$$

$$Z(x) = i\sqrt{\pi}e^{-x^2} - \sqrt{\pi}x\sum_{n=0}^{\infty}\frac{(-x^2)^n}{(n+1/2)!} \tag{19}$$

For complex arguments, see [*Fried and Conte, 1961*].

5 Plasma permittivity

The dielectric permittivity of hot (Maxwellian) plasma is

$$\epsilon(\omega, \mathbf{k}) = 1 + \sum_s \Delta\epsilon_s = 1 + \sum_s \frac{1}{k^2\lambda_s^2}G(x_s) \tag{20}$$

The summation is over charged species. For each species, we have introduced the Debye length

$$\lambda = \sqrt{\frac{\epsilon_0 T}{Nq^2}} = \frac{v}{\Pi}$$

where $v = \sqrt{T/m}$ is the thermal velocity and $\Pi = \sqrt{Nq^2/(m\epsilon_0)}$ is the plasma frequency; and

$$x = \frac{\omega - (\mathbf{k}\cdot\mathbf{u})}{\sqrt{2}kv}$$

where $\mathbf{u}$ is the species drift velocity.

For warm components, $x \gg 1$, we have

$$\Delta\epsilon = -\frac{\Pi^2}{[\omega - (\mathbf{k}\cdot\mathbf{u})]^2}\left(1 + \frac{3k^2v^2}{[\omega - (\mathbf{k}\cdot\mathbf{u})]^2}\right)$$

The dispersion relation for plasma oscillations is obtained by equating $\epsilon = 0$. For example, for warm electron plasma at rest,

$$\epsilon = 1 - \frac{\Pi^2}{\omega^2}\left(1 + \frac{3k^2v^2}{\omega^2}\right)$$

and we have the dispersion relation (for $\omega \approx \Pi$):

$$\omega^2 = \Pi^2 + 3k^2v^2 = \Pi^2 + k^2\langle v^2\rangle$$

6 Ion-sound waves

For electrons, $\omega \ll kv$ and for ions still $\omega \gg kV$. For $x \ll 1$, we use $G(x) \approx 1 + i\sqrt{\pi}x$:

$$\epsilon = 1 + \frac{\Pi_e^2}{k^2v^2}\left(1 + i\sqrt{\pi}\frac{\omega}{\sqrt{2}kv}\right) - \frac{\Pi_i^2}{\omega^2}\left(1 + \frac{3k^2V^2}{\omega^2}\right) \qquad (21)$$

where v and V are thermal velocities of electrons and ions, respectively. From $\epsilon = 0$ we have

$$1 + \frac{3k^2V^2}{\omega^2} = \frac{\omega^2}{k^2v_s^2}\left(1 + k^2\lambda^2 + i\sqrt{\frac{\pi}{2}}\frac{\omega}{kv}\right)$$

where $v_s = \sqrt{T_e/M}$. If we neglect V and imaginary part, then we get the usual relation $\omega = kv_s$.

References

[*Abramowitz and Stegun, 1965*] M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables", Dover, 1965.

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