Magneto-hydro-dynamic (MHD) waves

NGL

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1 MHD equations

1.1 Equations

In non-relativistic magnetohydrodynamics (MHD), the following equations are used. First, let us consider hydrodynamic equations:

1. Continuity equation
$$\dot{\rho} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}$$

2. Adiabatic state equation
$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) (p\rho^{-\gamma}) = 0 \tag{2}$$

   For homogeneous media, we can use just $p\rho^{-\gamma} = const$.

3. The Euler's equation (Newton's Second law per unit volume):
$$\rho \left(\dot{\mathbf{u}} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p + \mathbf{J} \times \mathbf{B} \tag{3}$$

   The last term is the Lorentz force, caused by presence of current $\mathbf{J}$ in the magnetic field $\mathbf{B}$.

Now, let us consider electrodynamic equations.

1. Ohm's law:
$$\mathbf{J} = \sigma \mathbf{E}' = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{4}$$

   where $\mathbf{E}'$ is the electric field in the system of the medium moving at non-relativistic speed $u \ll c$. The conductivity is assumed to be high enough so that $E \sim uB$.

2. Maxwell's equation #1. We assume that the medium is tenuous enough so that $\varepsilon = 1$, $\mu = 1$:

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\dot{\mathbf{E}} + \mu_0 \mathbf{J}$$

The first term is of the order $B(u/c)(v/c)/L$ and is neglected compared to B/L, where L is the typical length scale and $v \sim L/T$ is the typical wave speed, with T being the time scale. Thus,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{5}$$

3. Maxwell's equation #2:

$$\dot{\mathbf{B}} = -\nabla \times \mathbf{E} \tag{6}$$

1.2 Summary

From (4) and (5), $\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ and

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} - \mathbf{u} \times \mathbf{B} = \eta \nabla \times \mathbf{B} - \mathbf{u} \times \mathbf{B} \tag{7}$$

where we introduced $\eta = (\mu_0 \sigma)^{-1}$, the magnetic field diffusion coefficient. After substituting this into (6) and collecting all together:

$$\dot{\rho} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{8}$$

$$p\rho^{-\gamma} = const \tag{9}$$

$$\rho\left(\dot{\mathbf{u}} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) = -\nabla p + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \tag{10}$$

$$\dot{\mathbf{B}} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \tag{11}$$

where we used $\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$.

1.3 Frozen-in B

For infinite conductivity, $\eta = 0$. Equation (11), which now takes the form

$$\dot{\mathbf{B}} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

describes a frozen-in magnetic field, so that the magnetic field lines can be visualized to be moving together with the matter. This is proven by considering a magnetic flux

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$$

through an area S enclosed by contour C that is transformed together with the moving matter and showing that it is constant in time. Consider its change in time Δt:

$$\Delta\Phi = \Delta t \int_C \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{s} + \int_{S'=\Delta C} \mathbf{B} \cdot d\mathbf{s}$$

The first term can be transformed using

$$\int_S \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \oint_C \mathbf{B} \cdot (d\mathbf{l} \times \mathbf{u})$$

In the second term, the integration is over the surface that is created by the "slit" between C at t and the transformed contour at $t + \Delta t$. This term also can be transformed into integration over C using the area element $d\mathbf{s} = (\mathbf{u}\Delta t) \times d\mathbf{l}$ and then cancels the first term.

1.4 Energy flow

Using (7), the Poynting vector is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \eta \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{1}{\mu_0} \mathbf{u}_\perp B^2$$

where we used $\mathbf{u}_\perp B^2 = \mathbf{B} \times (\mathbf{u} \times \mathbf{B}) = \mathbf{u}B^2 - \mathbf{B}(\mathbf{u} \cdot \mathbf{B})$.

2 Linearized MHD equations

Consider an infinite uniform system, in which all physical quantities consist of a large part, denoted with index 0, which is constant (both in time and space), plus a small variable part (with index 1), e.g. $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$. The speed has only the small part $\mathbf{u} = \mathbf{u}_1$. We also assume infinite conductivity ($\eta = 0$), so that the waves are not attenuated. From the state equation (9) we get

$$p_1 = \rho_1 \gamma \frac{p_0}{\rho_0} = c_s^2 \rho_1$$

where c_s is the speed of sound. We use this to substitute p_1 everywhere in terms of ρ_1. The MHD equations (8,10–11) become after linearization

$$\begin{aligned}
\dot{\rho}_1 + \rho_0 \nabla \cdot \mathbf{u} &= 0 && (12)\\
\dot{\mathbf{B}}_1 &= \nabla \times (\mathbf{u} \times \mathbf{B}_0) && (13)\\
\rho_0 \dot{\mathbf{u}} &= -c_s^2 \nabla \rho_1 + \frac{1}{\mu_0}(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 && (14)
\end{aligned}$$

2.1 Sound

Disregard all electrodynamics. The linearized equations are

$$\begin{aligned}
\dot{\rho}_1 + \rho_0 \nabla \cdot \mathbf{u} &= 0\\
\rho_0 \dot{\mathbf{u}} &= -c_s^2 \nabla \rho_1
\end{aligned}$$

Taking the time derivative of the first equation, we get the wave equation

$$\ddot{\rho}_1 - c_s^2 \nabla^2 \rho_1 = 0$$

3 Plane waves

The plane waves are described by the harmonic time and space dependence

$$\rho_1 = \mathrm{Re}(\tilde{\rho} e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}})$$

etc. We use a tilde to denote the complex amplitude. The differential operators become (when they act on $\tilde{\rho}$ etc.) $\nabla \to i\mathbf{k}$ and $\partial/\partial t \to -i\omega$. The equations for complex amplitudes become

$$\begin{aligned}
\omega \tilde{\rho} &= \rho_0 \mathbf{k} \cdot \tilde{\mathbf{u}} && (15)\\
\omega \tilde{\mathbf{B}} &= -\mathbf{k} \times (\tilde{\mathbf{u}} \times \mathbf{B}_0) && (16)\\
\omega \rho_0 \tilde{\mathbf{u}} &= c_s^2 \mathbf{k} \tilde{\rho} - \frac{1}{\mu_0}(\mathbf{k} \times \tilde{\mathbf{B}}) \times \mathbf{B}_0 && (17)
\end{aligned}$$

Denote $\mathbf{n} = \mathbf{k}/\omega$:

$$\begin{aligned}
\tilde{\rho} &= \rho_0 \mathbf{n} \cdot \tilde{\mathbf{u}} && (18)\\
\tilde{\mathbf{B}} &= -\mathbf{n} \times (\tilde{\mathbf{u}} \times \mathbf{B}_0) && (19)\\
\rho_0 \tilde{\mathbf{u}} &= c_s^2 \mathbf{n} \tilde{\rho} - \frac{1}{\mu_0}(\mathbf{n} \times \tilde{\mathbf{B}}) \times \mathbf{B}_0 && (20)
\end{aligned}$$

Substituting the first and second equation into the third, we get

$$\tilde{\mathbf{u}} = c_s^2 \mathbf{n}(\mathbf{n} \cdot \tilde{\mathbf{u}}) + \frac{1}{\rho_0 \mu_0} (\mathbf{n} \times (\mathbf{n} \times (\tilde{\mathbf{u}} \times \mathbf{B}_0))) \times \mathbf{B}_0$$

Taking $\mathbf{B}_0$ along the z-axis, and introducing the Alfvén speed $v_A = B_0/\sqrt{\rho_0 \mu_0}$, and also dropping the tilde over $\mathbf{u}$, we have

$$\mathbf{u} = c_s^2 \mathbf{n}(\mathbf{n} \cdot \mathbf{u}) + v_A^2 \hat{z} \times (\mathbf{n} \times (\mathbf{n} \times (\hat{z} \times \mathbf{u}))) \tag{21}$$

The cross-products can be expanded to yield

$$\mathbf{u} = (c_s^2 + v_A^2) \mathbf{n}(\mathbf{n} \cdot \mathbf{u}) + v_A^2 \left(\mathbf{u} n_z^2 - \hat{z} n_z (\mathbf{n} \cdot \mathbf{u}) - \mathbf{n} n_z u_z \right) \tag{22}$$

where $n_z = \hat{z} \cdot \mathbf{n}$ and $u_z = \hat{z} \cdot \mathbf{u}$.

3.1 Alfvén waves

Consider $\mathbf{u} \perp \mathbf{n}$, so that $\mathbf{n} \cdot \mathbf{u} = 0$. Then (21) becomes

$$\mathbf{u} = v_A^2 \hat{z} \times (\mathbf{n} \times (\mathbf{n} \times (\hat{z} \times \mathbf{u})))$$

We see that we also have automatically $\mathbf{u} \perp \hat{z}$. Thus, the displacements are perpendicular to the plane containing $\mathbf{k}$ and $\mathbf{B}_0$. There is no density perturbation because $\tilde{\rho} = \rho_0 \mathbf{n} \cdot \mathbf{u} = 0$. From the second version (22) we get (taking into account that $u_z = 0$)

$$\mathbf{u} = v_A^2 \mathbf{u} n_z^2$$

from where the dispersion equation of Alfvén waves immediately follows:

$$1 = v_A^2 n_z^2$$

or

$$k_z = \pm \omega / v_A$$

Note that Alfvén wave, being a purely transverse (shear) wave, does not depend on the sound speed c_s at all.

3.2 Other waves

Let us split (22) into parts $\parallel \hat{z}$ and $\perp \hat{z}$, denoting $\mathbf{u} = u_z\hat{z} + \mathbf{u}_\perp$ and $\mathbf{n} = n_z\hat{z} + \mathbf{n}_\perp$. Notice that another way to write (22) is

$$\mathbf{u} - v_A^2 n_z^2 \mathbf{u}_\perp = c_s^2 \mathbf{n}(\mathbf{n}\cdot\mathbf{u}) + v_A^2 \mathbf{n}_\perp(\mathbf{n}_\perp \cdot \mathbf{u}_\perp)$$

The component of $\mathbf{u}$ which is perpendicular to both $\hat{z}$ and $\mathbf{n}$ is the Alfvén wave, and can be decoupled from other modes. (**NOTE:** One should check the logical validity of this statement.) Therefore, now let us assume that $\mathbf{u}$ is in the plane of $\mathbf{B}_0$ and $\mathbf{k}$, so that $\mathbf{u}_\perp \parallel \mathbf{n}_\perp$. Thus, $\mathbf{n}\cdot\mathbf{u} = n_z u_z + \mathbf{n}_\perp\cdot\mathbf{u}_\perp = n_z u_z + n_\perp u_\perp$, and we can again rewrite the equation for $\mathbf{u}$ as

$$\mathbf{u} - v_A^2 n^2 \mathbf{u}_\perp = c_s^2 \mathbf{n}(\mathbf{n}\cdot\mathbf{u})$$

The parallel part is

$$(1 - c_s^2 n_z^2) u_z = c_s^2 n_z n_\perp u_\perp \tag{23}$$

The perpendicular part is

$$(1 - v_A^2 n^2 - c_s^2 n_\perp^2) u_\perp = c_s^2 n_\perp n_z u_z \tag{24}$$

Substituting equations into each other we find that nontrivial solution is possible only when

$$(1 - c_s^2 n_z^2)(1 - v_A^2 n^2 - c_s^2 n_\perp^2) = c_s^4 n_\perp^2 n_z^2$$

or

$$c_s^2 v_A^2 n^4 \cos^2\theta - (c_s^2 + v_A^2) n^2 + 1 = 0 \tag{25}$$

where we used $n_z = n\cos\theta$. The solution of this equation gives the two remaining modes. Usually this equation is solved in terms of the phase velocity $v = 1/n$, the equation for which reads

$$v^4 - (c_s^2 + v_A^2) v^2 + c_s^2 v_A^2 \cos^2\theta = 0$$

with the solution

$$v_{1,2}^2 = \frac{1}{2}\left(c_s^2 + v_A^2 \pm \sqrt{c_s^4 + v_A^4 - 2c_s^2 v_A^2 \cos 2\theta}\right) \tag{26}$$

In the limiting case $c_s \ll v_A$ we get $v_1 \approx v_A$ (fast magnetoacoustic wave) and $v_2 \approx c_s \cos\theta$ (slow). In the opposite case ($c_s \gg v_A$), we have $v_1 \approx c_s$ (acoustic wave) and the slow wave is very similar to Alfvén wave, $v_2 \approx v_A \cos\theta$.

3.3 Polarization of plasma displacement of the magnetoacoustic waves

Using (24), (26), $1/n^2 = v^2$ and $n_\perp = n \sin\theta$, $n_z = n\cos\theta$, we get

$$\left(\frac{u_z}{u_\perp}\right)_{1,2} = \frac{1/n^2 - c_s^2 v_A^2 \sin^2\theta}{c_s^2 \cos\theta \sin\theta}$$

$$\left(\frac{u_z}{u_\perp}\right)_{1,2} = \frac{c_s^2 \cos 2\theta - v_A^2 \pm \sqrt{c_s^4 + v_A^4 - 2c_s^2 v_A^2 \cos 2\theta}}{c_s^2 \sin 2\theta} \tag{27}$$

From here, we immediately see that

$$\left(\frac{u_z}{u_\perp}\right)_1 \left(\frac{u_z}{u_\perp}\right)_2 = -1 \tag{28}$$

which means that the $\mathbf{u}_1 \perp \mathbf{u}_2$. Another expression for polarization can be obtained by solving (25) and substituting into (23), which leads to

$$\left(\frac{u_z}{u_\perp}\right)_{1,2} = -\tan\theta \frac{c_s^2 + v_A^2 \mp \sqrt{c_s^4 + v_A^4 - 2c_s^2 v_A^2 \cos 2\theta}}{c_s^2 - v_A^2 \mp \sqrt{c_s^4 + v_A^4 - 2c_s^2 v_A^2 \cos 2\theta}}$$

Using (28), we get formulas useful for $c_s > v_A$:

$$\left(\frac{u_z}{u_\perp}\right)_1 = \cot\theta \frac{c_s^2 - v_A^2 + \sqrt{c_s^4 + v_A^4 - 2c_s^2 v_A^2 \cos 2\theta}}{c_s^2 + v_A^2 + \sqrt{c_s^4 + v_A^4 - 2c_s^2 v_A^2 \cos 2\theta}} \tag{29}$$

$$\left(\frac{u_z}{u_\perp}\right)_2 = -\tan\theta \frac{c_s^2 + v_A^2 + \sqrt{c_s^4 + v_A^4 - 2c_s^2 v_A^2 \cos 2\theta}}{c_s^2 - v_A^2 + \sqrt{c_s^4 + v_A^4 - 2c_s^2 v_A^2 \cos 2\theta}} \tag{30}$$

This gives some insight into polarization in the limiting case $c_s \gg v_A$. We have $(u_z/u_\perp)_1 \approx \cot\theta$, $(u_z/u_\perp)_2 \approx -\tan\theta$, that is, $\mathbf{u}$ is approximately parallel to $\mathbf{k}$ for the fast (acoustic) wave, and perpendicular for the slow wave. The phase velocity of the slow wave is very similar to that of Alfvén wave, as obtained in the end of the previous Section.

Using (27), in the opposite limiting case ($c_s \ll v_A$) and intermediate range of angles (not too close to zero or $\pi/2$), we find that for the fast magnetoacoustic wave ($v_1 \approx v_A$) $\mathbf{u}_1 \perp \hat{z}$ and for the slow ($v_2 \approx c_s \cos\theta$) $\mathbf{u}_2 \parallel \hat{z}$.

Last remark: the polarization of the field of the wave $\mathbf{E}$, $\mathbf{B}_1$ is found from $\mathbf{E} = -\mathbf{u} \times \mathbf{B}_0$ and $\mathbf{B}_1 = \mathbf{n} \times \mathbf{E}$.