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Calculation of Electromagnetic Fields in Non-Planar Stratified Bi-Anisotropic and Non-Local Media

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***Abstract*—We present a method of calculation of monochromatic electromagnetic wave generation by electric and magnetic currents and propagation in a stratified bi-anisotropic medium. The method is stable against the "swamping" instability and has been generalized to (1) media with curved strata and (2) weakly non-local media such as warm plasma. This method has been implemented in MATLAB as StanfordFWM (Stanford Full Wave Method) code and successfully used to model propagation of radio waves in stratified ionospheric plasmas.**

I. Introduction

Stanford Full Wave Method (StanfordFWM) has been designed in papers [1], [2], [3] to calculate the propagation of electromagnetic waves in a planar-stratified anisotropic medium, with efficient use of parallelized computers. It is based on the idea of recursive calculation of reflection coefficients and mode amplitudes. The order of recursion is chosed so that the method is stable against the numerical "swamping" instability by evanescent waves [4, p. 574–576], which occurs when the evanescent wave solutions (with a large imaginary component of the vertical wave number) "swamp" the waves of interest. In this work, we enhance the StanfordFWM method by generalizing it to (1) media with curved strata and (2) weakly non-local media such as warm plasma.

II. Planar Stratified Local Bi-Anisotropic Medium

Consider a medium which is stratified in z-direction. Snell's law commands that transverse wave vector component $\mathbf{k}_\perp$ is conserved between layers. Let us find the propagation equation for the field components with a fixed $\mathbf{k}_\perp$ along z.

A. Propagation equation

A bi-anisotropic medium is charachterized by a constitutive 6×6 tensor

$$\hat{\mathsf{K}} = \begin{pmatrix} \hat{\varepsilon} & \hat{\xi} \\ \hat{\eta} & \hat{\mu} \end{pmatrix}$$

which relates $\mathbf{E},\mathbf{H}$ to $\mathbf{D},\mathbf{B}$ at the same spatial location (i.e., it is a local medium). Maxwell's equations may be written symbolically as

$$\frac{\hat{\mathsf{D}}}{ik_0}\mathbf{F} = \hat{\mathsf{K}}\mathbf{F} - \frac{\mathbf{J}}{ik_0}$$

where the 6×6 differential operator

$$\hat{\mathsf{D}} = \begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{pmatrix}$$

acts on the 6 field components

$$\mathbf{F} = \begin{pmatrix} \mathbf{E} \\ Z_0\mathbf{H} \end{pmatrix}$$

with 6 sources

$$\mathbf{J} = \begin{pmatrix} Z_0\mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix}$$

Perpendicular (x,y)-components $\mathbf{E}_\perp$, $\mathbf{H}_\perp$ are continuous even if the medium changes abruptly in z-direction, while z-components E_z, H_z may be discontinuous. Let us separate all fields into 4 "continuous" c-components

$$\mathbf{f} \equiv \mathbf{F}_c = \begin{pmatrix} \mathbf{E}_\perp \\ \mathbf{H}_\perp \end{pmatrix}$$

and 2 "discontinuous" or "dependent", parallel d-components

$$\mathbf{F}_d = \begin{pmatrix} E_z \\ H_z \end{pmatrix}$$

We will demonstrate that $\mathbf{f}$ are independent variables for propagation along z while $\mathbf{F}_d$ may be expressed in terms of $\mathbf{f}$.

The differential operator $\hat{\mathsf{D}}$ may be written as

$$\hat{\mathsf{D}} = \hat{\mathsf{U}}_x\partial_x + \hat{\mathsf{U}}_y\partial_y + \hat{\mathsf{U}}_z\partial_z$$

$\hat{\mathsf{U}}_i$ are constant matrices with the following properties:

$$\left.\begin{array}{rcccccc} (\hat{\mathsf{U}}_x)_{dd} &=& (\hat{\mathsf{U}}_y)_{dd} &=& (\hat{\mathsf{U}}_z)_{dd} &=& 0 \\ && (\hat{\mathsf{U}}_z)_{dc} &=& (\hat{\mathsf{U}}_z)_{cd} &=& 0 \\ && (\hat{\mathsf{U}}_x)_{cc} &=& (\hat{\mathsf{U}}_y)_{cc} &=& 0 \end{array}\right\} \quad (1)$$

Let us introduce

$$\begin{aligned}\hat{\mathsf{A}} &= \hat{\mathsf{U}}_x\frac{\partial_x}{ik_0}+\hat{\mathsf{U}}_y\frac{\partial_y}{ik_0}-\hat{\mathsf{K}}\\ \hat{\mathsf{B}} &= -\left[(\hat{\mathsf{U}}_z)_{cc}\right]^{-1}\\ \mathbf{j} &= \mathbf{J}_c+\hat{\mathsf{A}}_{cd}(\hat{\mathsf{K}}_{dd})^{-1}\mathbf{J}_d\end{aligned}$$

Using the properties of $\hat{\mathsf{U}}$ (1), we may readily express the d-components in terms of c-components:

$$\mathbf{F}_d=(\hat{\mathsf{K}}_{dd})^{-1}\hat{\mathsf{A}}_{dc}\mathbf{f}+(\hat{\mathsf{K}}_{dd})^{-1}\frac{\mathbf{J}_d}{ik_0}$$

and obtain the propagation equation along z in terms of c-components only:

$$\frac{\partial_z}{ik_0}\mathbf{f}=\hat{\mathsf{L}}\mathbf{f}+\frac{\hat{\mathsf{B}}\mathbf{j}}{ik_0}$$

where

$$\hat{\mathsf{L}}=\hat{\mathsf{B}}\left[\hat{\mathsf{A}}_{cd}(\hat{\mathsf{K}}_{dd})^{-1}\hat{\mathsf{A}}_{dc}-\hat{\mathsf{K}}_{cc}\right]$$

is the propagation operator [5], a 4×4 matrix dependent on $\hat{\mathsf{K}}$ and transverse derivatives $\nabla_\perp/(ik_0)$.

For a plane wave $\propto e^{i\mathbf{k}\cdot\mathbf{r}}$ and without sources we have

$$\frac{\nabla}{ik_0}=\frac{\mathbf{k}}{k_0}\equiv\mathbf{n},$$

the refractive index vector. $\hat{\mathsf{L}}$ becomes an algebraic operator represented as a constant matrix of size 4×4. The parallel refractive index component $q\equiv n_z$ is an eigenvalue of $\hat{\mathsf{L}}$:

$$\frac{\partial_z}{ik_0}\mathbf{f}=q\mathbf{f}=\hat{\mathsf{L}}\mathbf{f}$$

which is also called Booker equation for q [6]. Since $\hat{\mathsf{L}}$ is 4×4, there are 4 independent eigenwaves, with the parallel refractive indeces q_i and components $\mathbf{F}_i$ given by $(\mathbf{F}_i)_c=\mathbf{f}_i$ and $(\mathbf{F}_i)_d=(\hat{\mathsf{K}}_{dd})^{-1}\hat{\mathsf{A}}_{dc}\mathbf{f}_i$, $i=1\ldots 4$.

B. Recursive calculation of reflection coefficients and amplitudes

Thus, in a medium without sources we have found 4 different vertical normalized wavenumbers q and the corresponding 4 eigenmodes. We split them into 2 upward and 2 downward modes, according to the value of the imaginary part of q, so that the waves attenuate in the direction of their propagation. The upward and downward wave amplitudes form vectors $\mathbf{u}$ and $\mathbf{d}$, respectively. The reflection coefficients $\hat{\mathsf{R}}^{\mathrm{u}}$ and $\hat{\mathsf{R}}^{\mathrm{d}}$ are 2×2 matrices which convert upward and downward waves into each other, i.e., above the sources of the waves we have $\mathbf{d}=\hat{\mathsf{R}}^{\mathrm{u}}\mathbf{u}$ while below the sources $\mathbf{u}=\hat{\mathsf{R}}^{\mathrm{d}}\mathbf{d}$. The reflection coefficient from above, $\hat{\mathsf{R}}^{\mathrm{u}}$, is calculated starting at the upper boundary of the system (if the free boundary is assumed, there we have $\hat{\mathsf{R}}^{\mathrm{u}}=0$) and recursively going through layers downwards, using procedure outlined in [1]. The upward amplitude $\mathbf{u}$ is calculated starting at the lower boundary of the system and recursively going upwards. Traversing the boundaries between layers is done using the continuity of horizontal components of electric and magnetic fields $\mathbf{E}_\perp$ and $\mathbf{H}_\perp$. For $\hat{\mathsf{R}}^{\mathrm{d}}$ and $\mathbf{d}$, the direction of recursion is opposite. This direction of recursion ensures that StanfordFWM is stable against the “swamping” instability by evanescent waves, which is a problem for many similar methods.

C. Modeling of sources

The electromagnetic radiation may be emitted by an arbitrary configuration of harmonically varying currents (both electric and magnetic), which are modeled as flowing in the boundaries between layers and are included in the method as boundary conditions on electric and magnetic fields $\mathbf{E}_\perp$ and $\mathbf{H}_\perp$. Each boundary thus emits an upward wave (above the boundary) $\mathbf{u}^+$ and downward wave (below the boundary) $\mathbf{d}^-$ which are found from the jumps in the field $\Delta\mathbf{E}_\perp$ and $\Delta\mathbf{H}_\perp$ which are converted into $\Delta\mathbf{u}=\mathbf{u}^+-\mathbf{u}^-$ and $\Delta\mathbf{d}=\mathbf{d}^+-\mathbf{d}^-$, with $\mathbf{u}^\pm$ and $\mathbf{d}^\pm$ connected by $\hat{\mathsf{R}}^{\mathrm{u,d}}$. This procedure was also described in detail in [1].

D. Position space calculations

Independent calculation of waves with different $\mathbf{k}_\perp$ allows the code to be easily parallelized. The solution consisting of partial waves in the $\mathbf{k}_\perp$-domain is converted to the position $\mathbf{r}_\perp$-space by an inverse Fourier transform, thus providing a full wave 3D solution at any point in the medium. For problems involving propagation of radio waves in the Earth's ionosphere, there are sharp resonances in $\mathbf{k}_\perp$, which correspond to the Earth-ionosphere waveguide modes. In this case, to increase accuracy, it is better to work in a polar $\mathbf{k}_\perp$-grid which may be made denser at the resonances as implemented in [2].

III. Media with Curved Strata

We demonstrate, by considering a general curvilinear stratified system with anisotropic media, an approach to include the curvature into StanfordFWM. The results are validated for an isotropic medium in a cylindrically stratified system.

A. Modification of the propagation equation

Let us consider a coordinate system $\{\xi_1,\xi_2,\xi_3\}$ with scale factors $\{h_1,h_2,h_3\}$. The curl operation on an arbitrary vector field $\mathbf{A}$ is

$$\nabla\times\mathbf{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix}h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3\\ \partial_1 & \partial_2 & \partial_3\\ h_1A_1 & h_2A_2 & h_3A_3\end{vmatrix}$$

which may be written as a tensor

$$\hat{\mathsf{C}}=\nabla\times=\begin{pmatrix}0 & -\frac{1}{h_2h_3}\partial_3h_2 & \frac{1}{h_2h_3}\partial_2h_3\\ \frac{1}{h_1h_3}\partial_3h_1 & 0 & -\frac{1}{h_1h_3}\partial_1h_3\\ -\frac{1}{h_1h_2}\partial_2h_1 & \frac{1}{h_1h_2}\partial_1h_2 & 0\end{pmatrix}$$

The tensor $\hat{\mathsf{D}}$ may be separated into

$$\hat{\mathsf{D}}=\begin{pmatrix}0 & -\hat{\mathsf{C}}\\ \hat{\mathsf{C}} & 0\end{pmatrix}=\hat{\mathsf{D}}_1+\hat{\mathsf{D}}_2+\hat{\mathsf{D}}_3$$

where $\hat{\mathsf{D}}_i$ is only the part involving ∂_i.

Now, let us limit ourselves to a case when

$$h_{1,2}=1+\alpha_{1,2}\xi_3,\quad h_3=1,\quad \xi_3\equiv Z$$

so that $\alpha_{1,2} = const$ have the meaning of curvatures along coordinates $\xi_{1,2}$. E.g., after a cyclic permutation of the coordinate order, and starting at some radius R (such as the radius of the Earth) we have in cylindrical coordinates $Z = \rho - R$, $\alpha_1 = 1/R$, $\alpha_2 = 0$, and in spherical $Z = r - R$, $\alpha_1 = 1/R$, $\alpha_2 = 1/R$ (we scaled the angular coordinates so that they look like Cartesian at $r = R$). Let us look for equation for translation along the "radial" coordinate Z, in terms of "transverse" field components $\mathbf{F}_\perp$ along (ξ_1, ξ_2). The tensors D satisfy equations similar to (1). By analogy with the "flat" case considered above, let us introduce

$$\hat{\mathsf{A}} = \frac{\hat{\mathsf{D}}_1}{ik_0} + \frac{\hat{\mathsf{D}}_2}{ik_0} - \hat{\mathsf{K}}$$

Again, separating into c-components along $\xi_{1,2}$ and d-components along Z we can express the dependent components of the fields $\mathbf{F}_d$ in terms of continuous $\mathbf{f} \equiv \mathbf{F}_c$ as

$$\mathbf{F}_d = (\hat{\mathsf{K}}_{dd})^{-1}\hat{\mathsf{A}}_{dc}\mathbf{f}$$

and for $\mathbf{f}$ we have a propagation equation

$$-\frac{(\mathsf{D}_Z)_{cc}}{ik_0}\mathbf{f} = \left[\hat{\mathsf{A}}_{cd}(\hat{\mathsf{K}}_{dd})^{-1}\hat{\mathsf{A}}_{dc} - \hat{\mathsf{K}}_{cc}\right]\mathbf{f}$$

Let us substitute h_i into the above expression for $\hat{\mathsf{C}}$ and $\hat{\mathsf{D}}$. We then can express

$$\mathsf{D}_{1,2} = \frac{\partial_{1,2}}{h_{1,2}}\hat{\mathsf{U}}_{1,2}, \quad (\mathsf{D}_Z)_{cc} = -\hat{\mathsf{B}}^{-1}\left(\partial_Z + \frac{\alpha_1}{h_1}\hat{\mathsf{I}}_1 + \frac{\alpha_2}{h_2}\hat{\mathsf{I}}_2\right)$$

where $\hat{\mathsf{B}} = -[(\hat{\mathsf{U}}_Z)_{cc}]^{-1} = \hat{\mathsf{B}}^{-1}$ is the same as introduced before and $\hat{\mathsf{I}}_{1,2}$ are projection operators on $\xi_{1,2}$.

Thus, finally we have the propagation operator

$$\hat{\mathsf{L}} = \hat{\mathsf{L}}_{\text{flat}} - \frac{\alpha_1}{ik_0 h_1}\hat{\mathsf{I}}_1 - \frac{\alpha_2}{ik_0 h_2}\hat{\mathsf{I}}_2$$

defined as

$$\frac{\partial_Z}{ik_0}\mathbf{f} = \hat{\mathsf{L}}\mathbf{f}$$

The analog of the Booker equation is obtained when the fields are $\propto \exp(ik_0[\beta_1\xi_1 + \beta_2\xi_2])$, with $\beta_{1,2} = const$.

B. Validation

Let us consider a system consisting of an outside region of perfectly conducting cylinder of radius $a = 2$ filled with vacuum and a cylindrical source current sheet of radius $b = 2.5$ with uniform surface current parallel to the axis. The analytical solution may be obtained in cylindrical coordinates $\{\rho, \phi, z\}$, in terms of Bessel functions. For each fixed R, we can choose the curvilinear coordinates $\xi_1 = R\phi$, $\xi_2 = z$, $Z = \rho - R$. The field is axially symmetric (i.e., $\beta_1 = 0$). The full- wave method solution was obtained in the region $2 < \rho < 3$ with various stratum thicknesses $\Delta\rho$. Both analytic and StanfordFWM solutions are plotted in Fig. 1 for $k_0 = 4$, $\beta_2 = 0$ and show excellent agreement with each other, even for a coarse discretization $\Delta\rho = 0.1$. The maximum error for smaller $\Delta\rho$ was ~0.2% and was due to imperfect radiation conditions used in StanfordFWM at the outer boundary of $\rho = 3$. It is possible to demonstrate that the curvilinear solution is visibly different from the flat (Cartesian) geometry case.

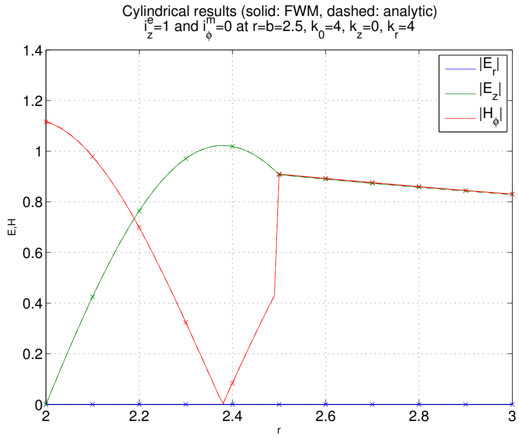

Fig. 1. StanfordFWM (solid) and analytic (dashed) solutions for the field in the system used in the validation example

IV. Warm Plasma (Non-Local Medium)

A. Generalization of the above analysis

In a hot plasma, the electrons may transfer information from one location to another, thus the electric displacement $\mathbf{D}$ at a given point in space depends not just on the value of the electric field $\mathbf{E}$ at the same location, but also on values of $\mathbf{E}$ in its vicinity. This makes the medium non-local, and introduces the spatial dispersion. Let us consider a warm plasma. Instead of Maxwell's equations, we have a system of hydro-electro-dynamic equations for the electromagnetic fields $\mathbf{E}$, $\mathbf{H}$, electron velocity $\mathbf{v}$, density n and pressure p:

$$\begin{aligned}
\nabla \times \mathbf{H} &= \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + qn\mathbf{v} \\
\nabla \times \mathbf{E} &= -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \\
pn^{-\gamma} &= const \\
m\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] &= \mathbf{G} \\
\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) &= 0
\end{aligned}$$

where $q = -e$ is the electron charge, $\gamma = 3$ is the adiabatic constant obtained from kinetic theory, and

$$\mathbf{G} = -\frac{\nabla p}{n} + q\left[\mathbf{E} + \mathbf{v}\times(\mathbf{B}_0 + \mu_0\mathbf{H})\right] - m\nu\mathbf{v}$$

is the force acting on electrons. We do not consider ions and are therefore constrained to the range of frequencies $\omega \gg \omega_{LH}$, the lower hybrid frequency.

Let us linearize this system for small disturbances $\propto e^{-i\omega t}$, i.e. $p = p_0 + p_1$, $n = n_0 + n_1$, $\mathbf{E} = \mathbf{E}_1$, $\mathbf{H} = \mathbf{H}_1$, $\mathbf{v} = \mathbf{v}_1$ with $n_0, p_0 = const$. In what follows, we dropped the subscript "1". We can express the density $n = p/(mc_s^2)$, where $c_s = \sqrt{\gamma p_0/(mn_0)}$ is the electron sound speed. For the rest of variables we have

$$\left.\begin{array}{rcl}\nabla\times\mathbf{H} & = & -i\varepsilon_0\omega\mathbf{E}+qn_0\mathbf{v}\\ \nabla\times\mathbf{E} & = & i\mu_0\omega\mathbf{H}\\ \nabla p/n_0 & = & i(\omega+i\nu)m\mathbf{v}+q\mathbf{E}+q\mathbf{v}\times\mathbf{B}_0\\ \nabla\cdot\mathbf{v} & = & i\omega p/(mc_s^2 n_0)\end{array}\right\}\quad (2)$$

Again, we may write these equations as

$$\frac{\hat{\mathsf{D}}}{ik_0}\mathbf{F}=\hat{\mathsf{K}}\mathbf{F}\quad (3)$$

where the field vector now includes, beside the electromagnetic field, also electron velocity and pressure, a total of 10 components:

$$\mathbf{F}=\begin{pmatrix}\mathbf{E}\\ Z_0\mathbf{H}\\ \frac{m\omega}{q}\mathbf{v}\\ \frac{Z_0 q}{m\omega}p\end{pmatrix}$$

The differential operator $\hat{\mathsf{D}}$ and constitutive tensor $\hat{\mathsf{K}}$ are now 10×10 matrices. Analogously to the bi-anisotropic local medium case, we separate all fields into the continuous c- and dependent d-components. Beside the perpendicular components of the electromagnetic field, also v_z and p are continuous. Thus, we have 6 independent c- and 4 d-components. The equations (1) are again valid and we can derive the propagation equation for $\mathbf{f}\equiv\mathbf{F}_c$. We generalize the method from Section II and calculate 3 upward and 3 downward wave amplitudes which are connected by 3×3 reflection coefficient matrices. An analogous approach was also implemented by [7], but they pointed out that “the problem of numerical swamping is severe”, while our algorithm is completely free from the swamping instability.

B. Validation by comparison to previous work

In order to validate our new method, we compare it to calculations done in previous works. In Figure 2, we calculated conversion on a plasma density ramp of electrostatic Langmuire waves (denoted as ES) into back-scatteredright-handed extraordinary (RX), left-handed ordinary (LO) and electrostatic (ES) modes, for conditions of Fig. 2 from [7]. The conversion efficiency is shown in dB as a function of the sine of the incidence angle (which is also a normalized perpendicular wave number $n_x=k_x/k_0$). We reproduced the features such as the effect of the so-called Ellis radio window at $n_x=0.520$.

Another comparison is done with the result of [8]. In Fig. 3, we recalculate the attenuation of the LO wave when reflected from an upward ramp in N_e for conditions of Fig. 10 of [8]. It is plotted as a function of a dimensionless factor p which was introduced in [8] and is proportional to the cube root of the plasma density gradient. We correctly reproduced the peaks and valleys which are obtained in [8] by a completely different method, namely contour integration in the complex k_z-plane.

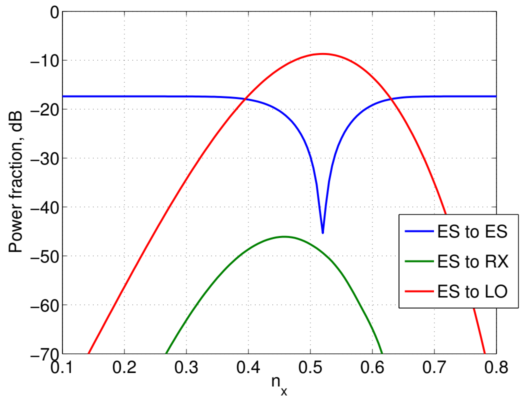

Fig. 2. The conversion efficiency of electrostatic (ES) waves on a plasma gradient for conditions in Fig. 2 of [7].

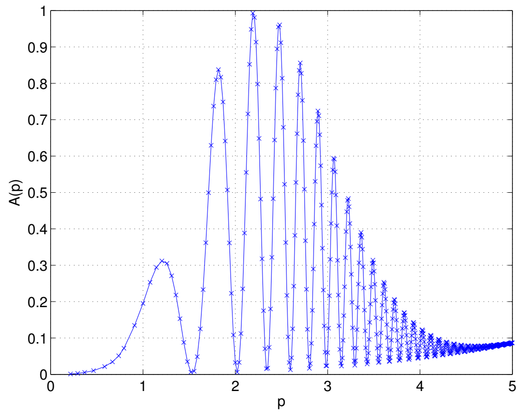

Fig. 3. Attenuation of LO wave as a function of the cube root of the plasma density gradient for conditions in Fig. 10 of [8].

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