Dynamics of a complex streamer structure

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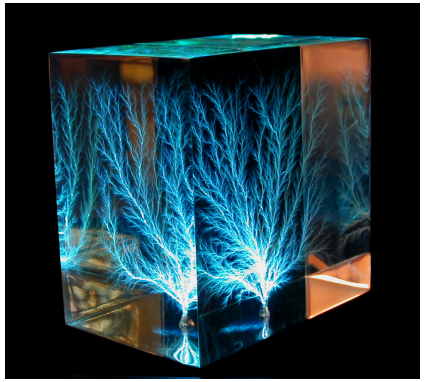
Outline

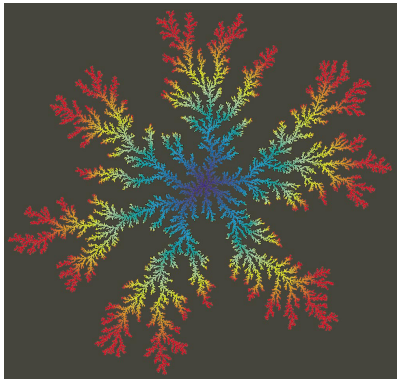
Fractal pattern in an electric discharge

Lichtenberg figure

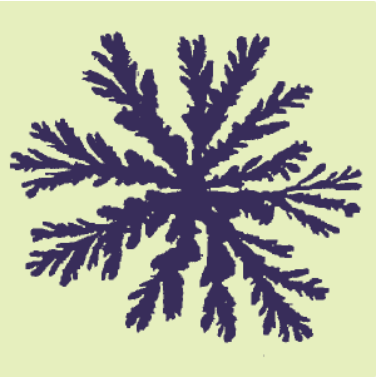


This is very similar to the following [Halsey, 2000]:

Diffusion-limited aggregation (DLA)



Hele-Shaw flow



All these have similar underlying math, which is usually named the DLA model.

Diffusion-limited aggregation (DLA) model

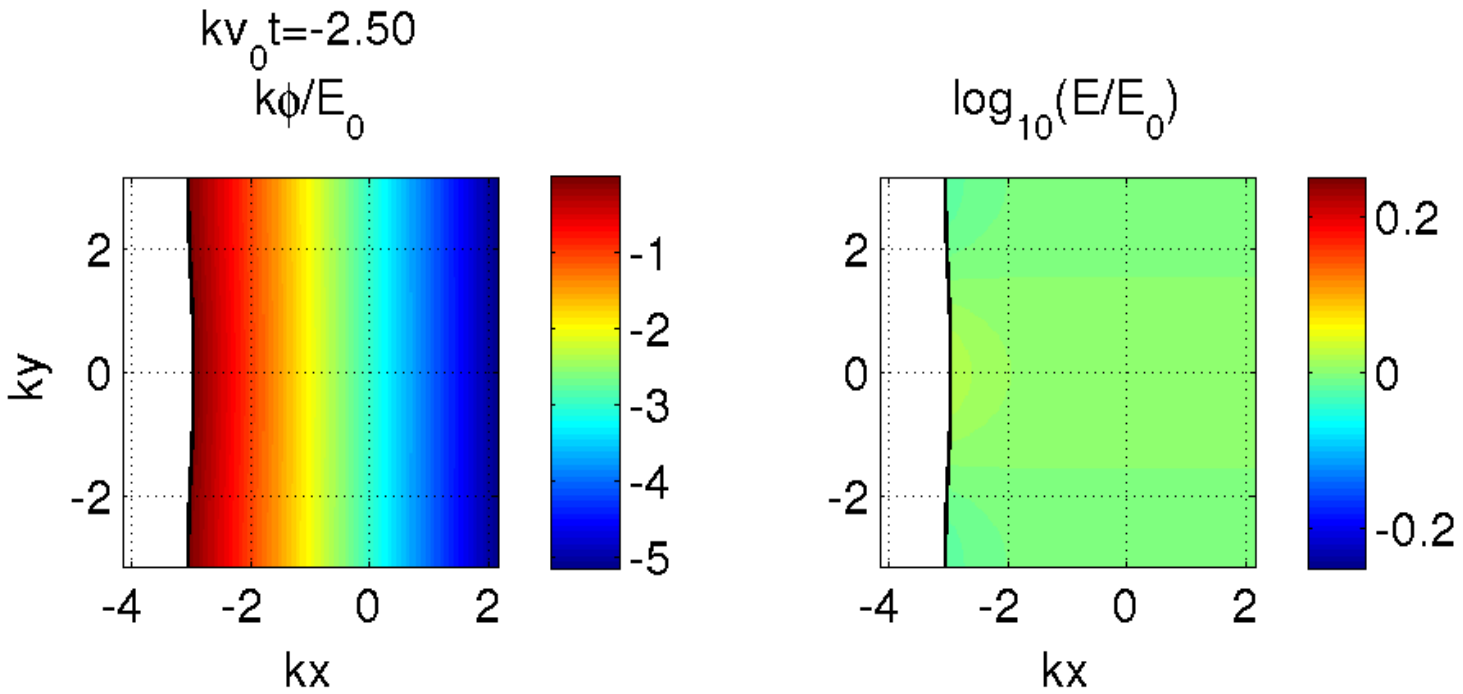
Consider a dynamic system made up from 2 media (A and B). A penetrates into B with velocity $\mathbf{v} \propto -\nabla p$, where p is defined in material B and is such that $\nabla^2 p = 0$ and $p = const$ at the A–B interface. Examples:

1. Electric discharge: A is a broken-down medium with high conductivity, B is a pre-breakdown dielectric, p is electrostatic potential, $\mathbf{E} = -\nabla p$ is electric field, the ionization front velocity $\mathbf{v} \propto \mathbf{E}$, $p = const$ in highly-conducting medium A.
2. DLA: p is the density of colloidal particles in B which quickly diffuse and attach to A (with flux $\propto \nabla p$; $p = 0$ at the interface).
3. Viscous fingering in Hele-Shaw flow [Saffman and Taylor, 1958]: A is an inviscid fluid (water), B is a viscous incompressible liquid in a porous medium (e.g., oil in sandstone), p is the pressure ($= const$ in A), velocity in B is $\mathbf{v} = -(k/\mu)\nabla p$, where k is the permeability in a porous medium and μ is the viscosity of the fluid; $\nabla \cdot \mathbf{v} = 0$.

Analytic solution in 2D

Initially straight front $\parallel \hat{y}$ propagating into uniform field $\mathbf{E} = E_0\hat{x}$; ; $\mathbf{v} = (v_0/E_0)\mathbf{E}$

k is the wavenumber of initial perturbation



Solution for the growth of a small harmonic perturbation is in terms of *curtate cycloids*. The field at the protrusions increases; perturbations sharpen until infinitely thin cusps are formed. Then, a fractal structure forms from infinitely thin protrusions, with branching at the preferred angle of 72° [Devauchelle et al., 2012] and fractal dimension $D \approx 1.67$–1.71 [Halsey, 2000].

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Goal

To verify and/or amend the relation $\mathbf{v} \propto \mathbf{E}$ which was assumed for the DLA system.

Quasi-electrostatic (QES) equations

We use QES equations [Pasko et al., 1997] with constant electron mobility $\mu < 0$ (ν is the net ionization $\nu_i - \nu_a$):

$$\left.\begin{aligned} \mathbf{E} &= -\nabla\phi \\ \nabla\cdot\mathbf{E} &= \rho \\ \dot{\rho} &= -\nabla\cdot(\sigma\mathbf{E}) \\ \dot{\sigma} &= \nu(|\mathbf{E}|)\sigma \end{aligned}\right\}$$

It may be shown that this system cannot describe streamers: this is done by spatial rescaling and showing that there is no intrinsic spatial scale. The streamer mechanisms are needed for propagation:

1. Electron drift adds $\nabla\cdot(\mu\mathbf{E}\sigma)$ to the LHS of the last equation.
2. Electron diffusion adds $D\nabla^2\sigma$ to the RHS.
3. Photoionization adds an extra source p to the RHS. It is non-local, $\propto S_i = \nu\sigma$ at a distance:

$$p(\mathbf{r}) = \int S_i(\mathbf{r}')F(\mathbf{r}-\mathbf{r}')\,d^3\mathbf{r}', \text{where } F(\mathbf{r}) = F(r) \to 0 \text{ for } r \to \infty$$

Ionization front

Solve in 1D for an ionization front with curvature $\varkappa$ propagating with a constant velocity v along axis x

from S : streamer (or broken-down, ionized) region at $x \to -\infty$

into N : neutral (or pre-breakdown, non-ionized, , i.e., $\sigma = 0$) region at $x \to +\infty$, with given external electric field $\mathbf{E}(+\infty) = \mathbf{E}_0 = \hat{x}E_0$

Let us obtain the value (or range of values) for the velocity v which satisfies

1. finiteness;
2. correct boundary conditions at $x = +\infty$ (i.e., in N);
3. physical value restraints, such as $\sigma > 0$.

Ionization front determined by photoionization

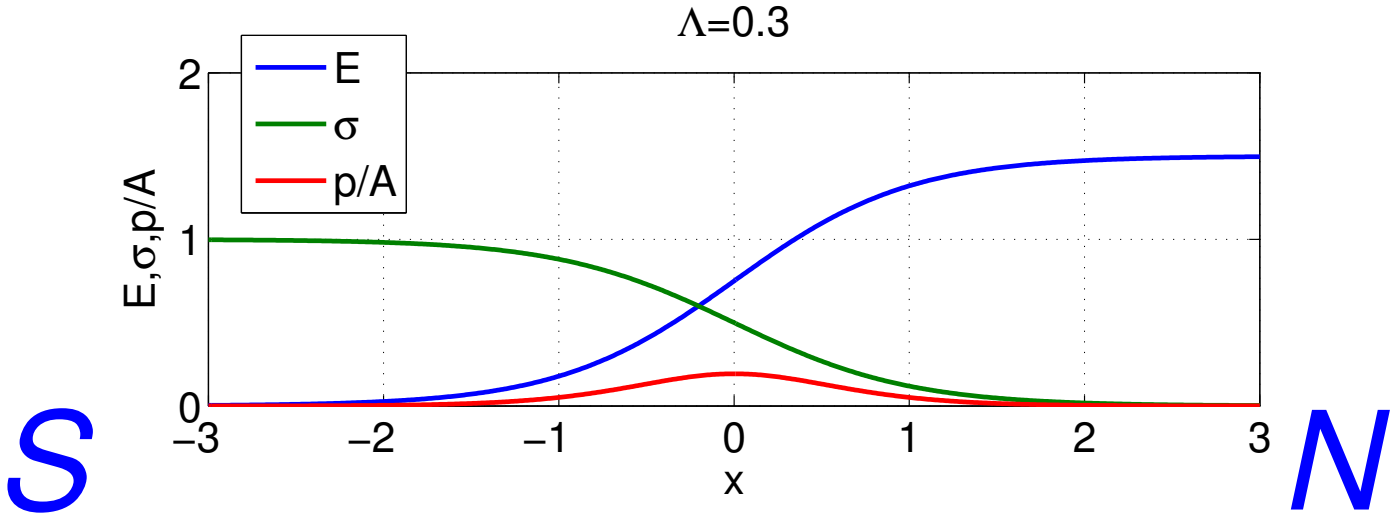
The photoionization is a non-local source $p(\mathbf{r}) = \int S_i(\mathbf{r}')F(\mathbf{r}-\mathbf{r}')\,d^3\mathbf{r}'$ proportional to the "local" ionization rate $S_i = \nu(E)\sigma$.

Consider a simple "exponential profile" model [Luque et al., 2007]:

$$F(r) = \frac{A}{\Lambda^2}\frac{e^{-r/\Lambda}}{4\pi r}$$

where Λ is the "length" and $A = \int F(\mathbf{r})d^3\mathbf{r} \ll 1$ is the "strength" (of photoionization). The ionization front looks like this:

$\nu(E) = \beta E$, $\mu = 0$, $\varkappa = 0$, $A \to 0$, $v = v_s = \Lambda\beta E_0$



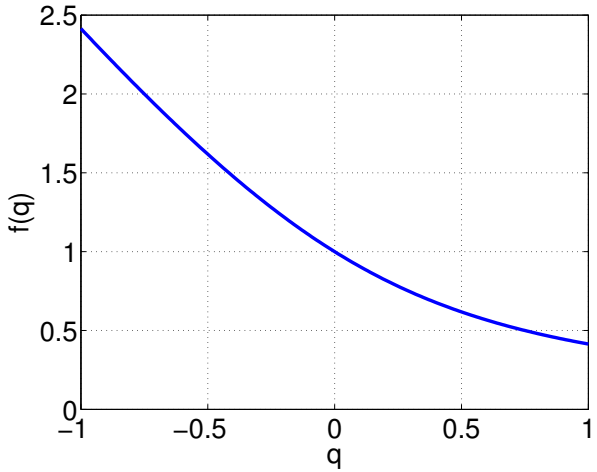
Ionization front determined by photoionization (2)

The front velocity is found to be in the range

$$v > v_s = \Lambda \nu(E_0) f(q) \left[1 + O\left(\sqrt{2A}\right)\right]$$

where $f(q)$ with $q = \varkappa\Lambda/2$ contains curvature dependence.

$$f(q) = \sqrt{1+q^2} - q$$



Observations:

1. The "strength" $A \ll 1$ plays only a minor role in determining the velocity, Λ has a greater role (paradox for $A \to 0$)
2. Curvature *lowers* the velocity $\propto f(q)$. Intuitive understanding: the photoionization in a convex front not as efficient because photons are scattered out. The most efficient growth of perturbations is at scales $\sim \Lambda$.
3. The minimum ionization front thickness $\approx \Lambda$ corresponds to minimum velocity $v = v_s$. This is the "selected front" [Arrayás and Ebert, 2004].

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Relation of the ionization front velocity result to DLA

- Reminder: in a DLA system which creates a fractal pattern, the front propagates with velocity $\mathbf{v} \propto \mathbf{E}$.
- We get an "almost DLA" system in the photoionization case, for $\nu = \beta\,|\mathbf{E}|$ ($\beta > 0$):

$$\mathbf{v} = \pm\Lambda\beta f(q)\mathbf{E}, \quad q = \varkappa\Lambda/2, \quad f(q) = \sqrt{1+q^2} - q$$

where $\pm$ corresponds to the polarity of the streamer. The dependence $f(q)$ must determine the transverse size of streamers.

Previous modeling of a fractal electric discharge [Niemeyer et al., 1984, 2D]

- The velocity is modeled as cluster growth probability $P \propto E^{\eta}$
- The DLA system is for $\eta = 1$ but other values also give fractal structures

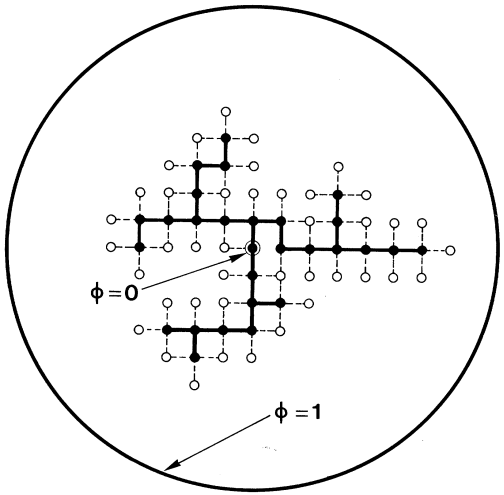


FIG. 2. Illustration of the stochastic model we introduce to simulate dielectric breakdown on a lattice. The central point represents one of the electrodes while the other electrode is modeled as a circle at large enough distance. The discharge pattern is indicated by the black dots connected with thick lines and it is considered equipotential ($\phi = 0$). The dashed bonds indicate all the possible growth processes. The probability for each of these processes is proportional to the local electric field (see text).

TABLE I. Dependence of the Hausdorff dimension D on the exponent η used in the relation between probability and local field [Eq. (3)].

η	D
0	2
0.5	1.89 ± 0.01
1	1.75 ± 0.02
2	$\sim 1.6^{a}$

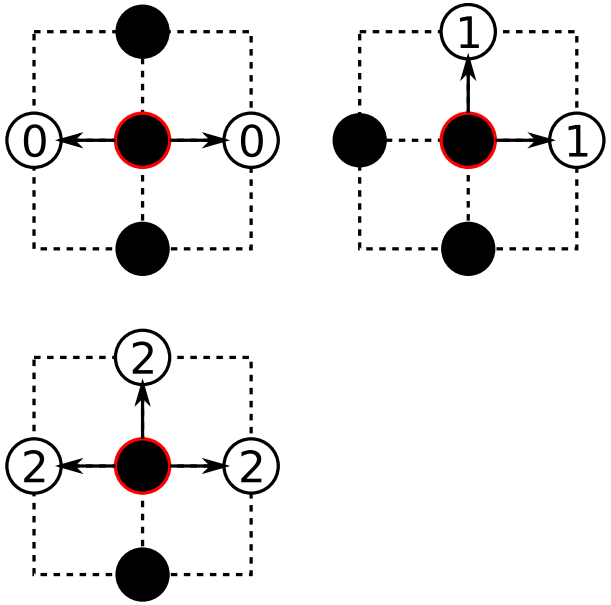
Latest theory: $D_{\mathrm{DLA}} \approx 1.67$–$1.71$ in 2D [Halsey, 2000].

Curvature on a dicrete 2D grid

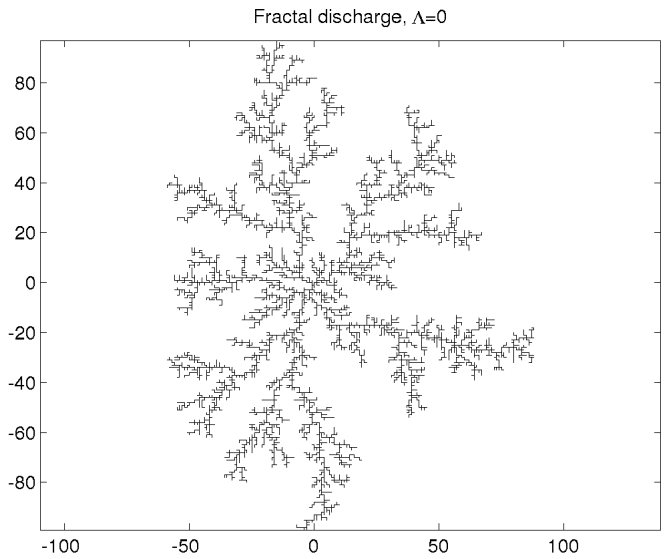
Including curvature on a discrete grid is not very accurate. We approximate the notion of curvature with the number indicating into how many directions (next to the chosen direction) the cluster can grow at a given point.

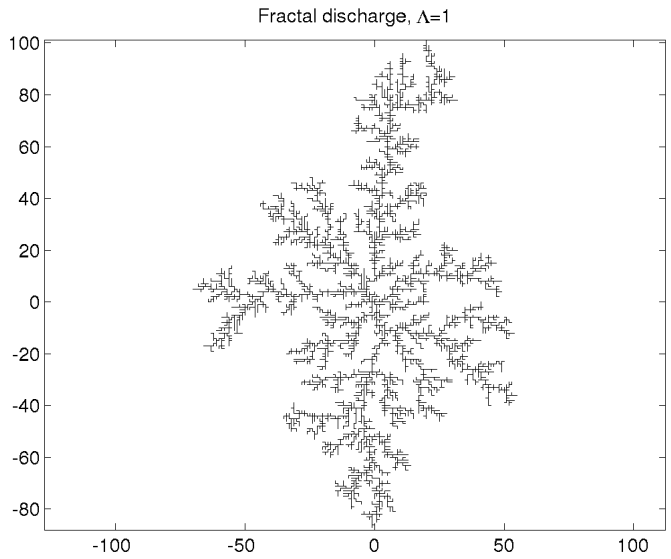
Here is how we define the curvature (in units of $1/a$ where $a = 1$ is the grid step):

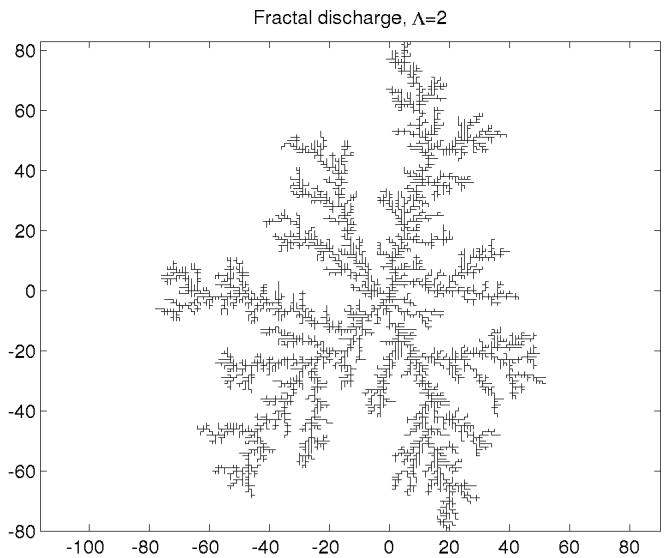
1. a "flat" surface (line) gives $\varkappa = 0$;
2. a "corner" gives $\varkappa = 1$;
3. a "rod" gives $\varkappa = 2$;

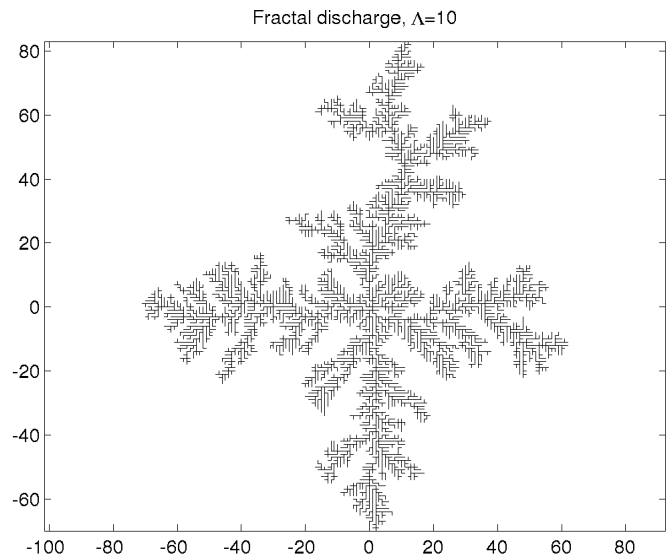


Results for varying photoionization length Λ









Outline

Summary

- Ionization front velocity was calculated for electron drift, electron diffusion and photoionization streamer mechanisms. Most of the results were not shown, see next slide!
  The most interesting results:
  1. A range of velocities $v > v_s$ is always obtained instead of a fixed number;
  2. Finite $v_s = v_{min}$ for infinitely small photoionization strength A (paradox!).

  This suggests that the streamer velocity may fluctuate significantly even for small changes in the parameters of the model.

- Fractal modeling result: The transverse size of the simulated streamer is of the order of the photoionization length Λ. The analysis of small harmonic perturbations of a flat ionization front gives the same result.

Ionization front velocity, all mechanisms

Notation: $\nu_d = \nu_0 - \varkappa\mu E_0$, $\nu_0 = \nu(E_0)$

1. No streamer mechanisms: $v_s = 0$ (no propagation).
2. Electron drift (with mobility $\mu < 0$): $v_s = \mu E_0$ for negative streamers ($E_0 < 0$), $v_s = 0$ for positive streamers (no propagation).
3. Electron drift + diffusion with coefficient $D = const$:

$$v_s = \mu E_0 + 2\sqrt{D\nu_d} - \varkappa D$$

   For $\varkappa = 0$ this is same as Ebert et al. [1997].
4. Electron drift + photoionization with length Λ:

$$v_s \approx \mu E_0 + \Lambda\nu_d f(q), \quad q = \frac{\varkappa\Lambda}{2}, \quad f(q) = \sqrt{1+q^2} - q$$

Comments:

- If formula gives $v_s < 0$, must take $v_s = 0$.
- Solutions exist for $v > v_s$, but σ is minimal in the front of the streamer at $v = v_s$, this is minimal advanced ionization (MAI) condition, corresponding to the "selected front" of Arrayás and Ebert [2004].
- The velocity is generally reduced for a convex ($\varkappa > 0$) front.

Outline

References

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